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Inventory Control with Limited Capacity and Advance Demand Information

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Manufacturers make production decisions and carry inventory to satisfy uncertain demand. When holding and shortage costs are high, carrying inventory could be even more expensive for a *capacitated* production system. Recent developments in information technology and sales strategies enabled firms to acquire, collect, or induce advance demand information. We address a periodic-review, stochastic, capacitated, finite and infinite horizon production system faced by a manufacturer who has the ability to obtain advance demand information. We establish optimal policies and characterize their behavior with respect to capacity, fixed costs, advance demand information, and the planning horizon. With a numerical study, we quantify the value of advance demand information and additional capacity for specific problem instances. We illustrate how advance demand information can be a substitute for capacity and inventory.

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1. Introduction

Advance demand information for a product is obtained when customers place orders in advance for a future delivery. Under advance demand information models, the total demand to be realized at a future period s is given by the sum of orders received prior to current period $t < s$ and the orders that are going to be received thereafter. A recent example that illustrates our model is Dell's "Intelligent Fulfillment," which allows for four different levels of response time to customers: standard (conventional: five-day promised order lead time), value delivery (slower: lower shipping cost), premium delivery (faster: next day delivery), and precision delivery (specific date). This broader choice for delivery mode and timing will presumably enhance Dell's interaction with its customers.[1]

Companies could offer, for example, price discounts to entice customers to accept longer response time (Chen 2001), which would essentially lead to a mix of build-to-order and build-to-stock production strategy and reduce inventory levels and related costs. Capacity expansion quite often requires interruption of production runs. On the other hand, a portfolio of customers with different required response times could lead to better capacity utilization, consistent revenues, and reduced inventory management costs if the manufacturer uses this information properly. In this paper, we study the resulting *capacitated* production problem and assess the joint role and value of advance demand information and production capacity. Understanding this relationship enables a production manager to design cost effective incentives to induce advance orders and to quantify the impact of additional capacity. To do so, we first establish effective production policies and characterize their behavior. Then, we show that advance demand information not only helps reduce inventories and use the available capacity efficiently, but also enables production policies that are responsive to the changes in demand.

We can specify the demand model for the above Dell example as $D_t = (D_{t,t}, \ldots, D_{t,t+N})$, where $D_{t,s}$ represents orders placed during period t for future periods $s \in \{t, \ldots, t+N\}$. N is the length of the information horizon. We assume linear production costs and exogenously specified fixed production lead time. In each period production cannot exceed a fixed capacity. Demand is satisfied through on-hand inventory, if any. Otherwise, unsatisfied demand is fully backordered. All other costs, in particular holding and penalty costs, are expressed as a function of a *modified* inventory position, which is defined as inventory on hand plus outstanding orders minus backorders minus part of the early commitments. The state of the system is given by this modified inventory position and observed demands beyond the production lead time. The manufacturer's objective is to minimize the expected discounted cost of managing the capacitated system over a finite or an infinite horizon.

For problems with *zero* fixed costs, we prove the optimality of a state-dependent *modified* base stock policy under the assumption that the single period expected cost is convex in the modified inventory position. At each period the policy triggers a production order to increase the modified inventory position to a level as close as possible to that period's base stock level. For problems with *positive* fixed costs, we establish the optimality of a state-dependent *threshold* policy when we restrict the production decision to either full capacity or nothing. Under this policy producing the full capacity is optimal if the modified inventory position is below the threshold. Otherwise, it is optimal not to produce anything. In process industries such as oil and gas, sugar refining, and steel or aluminum hot roll pressing, high fixed costs commonly restrict the policy space to a production decision of either full capacity or nothing (see Schroder 1993, Chapter 6). We establish the monotonicity of the base stock level, the threshold policy, and the intertemporal cost function with respect to capacity, advance demand information, fixed cost, and the number of remaining periods in the planning horizon. We show, for example, that the base stock level and the threshold are increasing[2] functions of observed demand beyond the lead time. In contrast to the *zero* fixed cost problem, we illustrate through a numerical example that increasing the capacity does not necessarily reduce inventory and related costs when fixed cost is positive and the policy is to produce either full capacity or nothing.

While the structural results show how the system responds to changes in key parameters for *all* problem instances, a numerical study quantifies the system's response for specific problem instances. Hence, we conclude the paper with a numerical study and address issues that may arise in the design of capacitated production systems. Through this study we also verify our structural results. We illustrate that advance demand information is a substitute both for inventory and capacity. Systems that can obtain advance demand information have lower inventory levels and related costs than the classical capacitated systems. We also observe diminishing returns both to additional capacity and advance demand information. The model with numerical experiments can help develop effective design strategies for planning capacity, obtaining advance demand information, and reducing fixed costs.

1.1. Literature Review

Early works for *zero* fixed cost inventory models that generalize demand to account for seasonal variations and nonstationary data[3] are primarily due to Karlin (1960), Veinott (1965), and Zipkin (1989). They extend the classical Arrow-Harris-Marchak dynamic inventory model and prove the optimality of a period dependent base stock policy, in which the demand distribution changes from period to period. Veinott (1965), for example, establishes the optimality of a myopic policy when demand distributions are stochastically increasing over time. In a later study, both for finite horizon and infinite horizon problems, Federgruen and Zipkin (1986) address the *capacity* constraint for *stationary* inventory problems and prove the optimality of a modified base stock policy.

Aviv and Federgruen (1997) and Kapuscinski and Tayur (1998) combine the above stream of research and address the capacitated single-item inventory control problem with periodic demand pattern. They establish the optimality of a period-dependent modified base stock policy. Aviv and Federgruen provide a value iteration method; whereas Kapuscinski and Tayur employ a simulation-based optimization method to compute modified base stock levels and to provide insights into the joint role of periodic demand and capacity.

Several researchers, noticing the dynamic nature of demand, incorporate the forecast update process to inventory problems. Güllü (1996), for example, studies the optimal policy under a martingale model for forecast evolution (MMFE) for a capacitated single-item periodic review inventory system with *zero* lead time and *zero* fixed costs. To the best of our knowledge, Hausman (1969) introduces this forecast evolution model. Heath and Jackson (1994) generalized his model, coined the term MMFE, and used this model in a simulation study. Toktay and Wein (2001) model a production system as single-server discrete-time continuous state queue under MMFE. Note that forecast updates could result from several factors such as promotions, advertisement, or market movements. By focusing on advance demand information, we characterize the optimal policy under this information, quantify its value, and provide a means to find efficient incentives to induce advance orders (as in Chen 2001).

In the case of *positive* fixed costs, even in the absence of advance demand information, the form of the optimal policy for capacitated systems is unknown. Gallego and Scheller-Wolf (2000) partially characterize the optimal policy and extend the work of Shaoxiang and Lambrecht (1996). In a technical note Gallego and Toktay (1999) address the same problem as in Güllü (1996) but with a positive fixed cost and they establish the optimality of a "bang-bang" policy when the policy space is restricted to produce either the full capacity or nothing. In addition to the optimality of a threshold policy, we establish the monotonicity of the threshold with respect to the fixed cost, advance demand information, and the remaining periods in the planning horizon. We then address the infinite horizon problem.

For an *uncapacitated* problem, Song and Zipkin (1993) and Sethi and Cheng (1997) incorporate Markov modulated demand information structure and prove the optimality of state dependent policies. Gallego and Özer (2001) study the uncapacitated version of this paper. They establish the optimality of state dependent base stock and (s, S) policies and summarize the literature for uncapacitated systems. We provide a connection to their paper through Corollary 1. The concept of demand lead times was first introduced in continuous review problems by Hariharan and Zipkin (1995). In their model, all customers place orders l time units in advance. Hence, the manager has perfect information about the future demand. They prove the optimality of a base stock policy by showing that the demand lead times offset the supply lead times. Marklund (2002) extends their work to one-warehouse multiretailer distribution systems. Schwarz et al. (1998) attempt to generalize the Hariharan and Zipkin model to account for imperfect demand information. Claiming that an optimal policy would be complex, Dellaert and Melo (2002) provide a heuristic to solve an *uncapacitated stationary* inventory problem under partial demand information. The capacity constraint poses significant challenges and, hence, has attracted many researchers. We hope that our structural results, together with the numerical study, contribute to the general understanding of capacitated problems both for zero and positive fixed cost inventory problems and bring the literature one step closer to a more general inventory model.

The remainder of this paper is organized as follows. In §2, we introduce the notation and the model of advance demand information. In §3, we establish the optimal policy and structural results for inventory problems with zero fixed costs. In §4, we present the results for inventory problems with positive fixed costs. In §5, we extend the results for the α-discounted infinite horizon problem. In §6, we offer managerial insights through a numerical study. In §7, we conclude the paper.

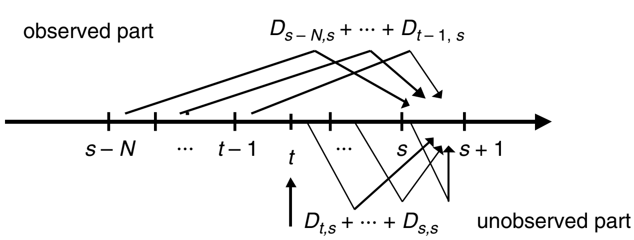
2. Model Formulation

During each period t, an arriving customer either purchases the product in period t, or places an order for delivery in a future period $s \in \{t+1, \ldots, t+N\}$. From the perspective of the production manager, the demand stream during period t is a vector: $D_t = (D_{t,t}, \ldots, D_{t,t+N})$, where $D_{t,s}$ represents the nonnegative demand for period s placed during period t and $N < \infty$ is the length of the *information horizon*.

Under this demand model, at the beginning of each period t, we can decompose the demand for period $s \geqslant t$ into two parts as illustrated in Figure 1: the *observed* part $O_{t,s} \equiv \sum_{r=s-N}^{t-1} D_{r,s}$ and the *unobserved* part $U_{t,s} \equiv \sum_{r=t}^{s} D_{r,s}$.

The sequence of events at the beginning of period t is as follows. (1) The manager reviews on-hand inventory I_t, backorders B_t, the pipeline inventory, and the observed part of the demand for periods $s \in \{t, t+1, \ldots, t+N-1\}$.

Figure 1. Observed and unobserved part of the demand.



She decides whether or not to produce $z_t \geqslant 0$, which cannot exceed Q units due to limited capacity. She incurs a *nonstationary* production cost of $K_t\delta(z_t) + c_t z_t$, where $\delta(z) = 1$ if $z > 0$, K_t is the fixed production cost, and c_t is the variable production cost. (2) The production initiated at period $t-L$ is added to the inventory, that is, L periods are required to complete the production. (3) The demand vector D_t is observed. The demand for period t is satisfied through on-hand inventory; otherwise it is backordered. (4) The manager incurs holding and penalty costs based on end-of-period net inventory.

Completing production takes L periods; hence, the manager should protect the system against the lead time demand, which is the total demand realized during periods $\{t, t+1, \ldots, t+L\}$. Due to advance demand information, she knows part of the lead time demand, that is, $\sum_{s=t}^{t+L} O_{t,s}$. The expected cost charged to period t is based on the net inventory at the end of period $t+L$. Let

x_t: modified inventory position *before* production decision $= I_t + \sum_{s=t-L}^{t-1} z_s - B_t - \sum_{s=t}^{t+L} O_{t,s}$,
y_t: modified inventory position *after* production decision $= x_t + z_t$.

Note that these variables subtract the observed part of the lead time demand. In addition to x_t, we also keep track of observations beyond the lead time, $O_t = (O_{t,t+L+1}, \ldots, O_{t,t+N-1})$. At the end of the period t, we update the state space by

$$x_{t+1} = x_t + z_t - D_{t,t} - \sum_{s=t+1}^{t+L+1} D_{t,s} - O_{t,t+L+1}, \tag{1}$$

$$O_{t+1} = \big(O_{t,t+L+2} + D_{t,t+L+2}, \ldots, O_{t,t+N} + D_{t,t+N}\big). \tag{2}$$

The expected holding and penalty costs charged to period t is given by $\widetilde{G}_t(y_t) = \alpha^L E g_{t+L}(y_t - \sum_{s=t}^{t+L} U_{t,s})$, where α is the discount factor and the expectation is with respect to the unobserved part of the lead time demand. We assume that g_t is convex and coercive for all t. We call a function $g: \mathscr{R} \to \mathscr{R}$ coercive if $\lim_{|x|\to\infty} g(x) = \infty$. These properties are satisfied, for example, when holding and penalty costs are linear. We also assume that leftover inventory at the end of the planning horizon T is salvaged for c_{T+1} per item. The solution to the following dynamic programming recursion minimizes the cost of managing this capacitated system for a finite horizon problem with $T-t$ periods remaining until termination:

$$J_t(x_t, O_t) = \min_{x_t \leqslant y_t \leqslant x_t + Q} \big\{K_t\delta(y_t - x_t) + V_t(y_t, O_t)\big\}, \tag{3}$$

$$V_t(y_t, O_t) = G_t(y_t) + \alpha E J_{t+1}(x_{t+1}, O_{t+1}), \tag{4}$$

where $J_{T+1}(\cdot,\cdot) \equiv 0$ and $G_j(y_j) = (c_j - \alpha c_{j+1}) y_j + \widetilde{G}_j(y_j)$. The function $J_t(x_t, O_t)$ is the optimal cost of managing this system when the initial state space is (x_t, O_t) and there are $T-t$ periods remaining. The function $V_t(\cdot)$ is the auxiliary cost function. The formal construction of this dynamic program and the conversion to a problem with zero salvage value is similar to that in Gallego and Özer (2001).

We assume that the vector D_t is independent across time but not necessarily stationary. Hence, the unobserved part of the demand is independent of the observed part. This demand scenario might exist, for example, when the manufacturer segments the customer with respect to the existing relationships, perhaps by quoting a shorter lead time to a more valued customer. Each element of D_t is then independent across time, just as in the classical case, although they are allowed to be correlated within D_t. We assume that advance orders are firm. Cancellations and early fulfillment of demand are not allowed, as might be the case, for example, when manufacturers have long-term price agreements with customers for advance orders. Breaching a contract may result in severe penalty costs. Our analysis does not require differentiability and applies to both continuous and discrete demand cases.

Before we proceed to the analysis of the problem, we deal with two trivial cases here. (1) The problem with $N \leqslant L+1$: then the state space collapses into a single dimension. For this case the classical results for capacitated systems with stationary data, such as in Federgruen and Zipkin (1986), would apply as long as the inventory position is *modified* to account for observed demands. (2) The problems with *unlimited* capacity, *stationary* cost, and demand distributions and *zero* fixed cost: for this case a myopic approach of decomposing the production control problem to deal with stochastic (unobserved) and deterministic (observed) part of the demand independent of each other would be optimal. Careful thought reveals that observed demand information beyond the lead time, $(O_{t,t+L+1}, \ldots, O_{t,t+N-1})$, does not affect the production system. Nevertheless, we show this in Corollary 1. Classical results would apply here as well. From the definition of modified inventory position x_t, an increase in $\sum_{s=t}^{t+L} O_{t,s}$, the observed part of the lead time demand, would clearly increase production in the current period. From this point on we consider the more general case of *capacitated* production systems with $N > L+1$ and *nonstationary* data. When O_t is known deterministically, we refer to it as o_t and suppress the subscript t whenever doing so does not cause confusion.

3. Analysis for Zero Fixed Costs

The dynamic programming recursion for the zero fixed cost problem is given by Equation (3) with $K_t = 0$. We prove the optimality of a state-dependent modified base stock policy, which is characterized by a single critical level: If the modified inventory position at the beginning of period t before production decision is below this critical level, the manufacturer should produce enough to bring it up to the critical level, or as close to it as possible. Otherwise, she should not produce. The following theorem establishes the optimal policy. Note that some of the proofs are provided in the appendix. For a complete discussion we also refer the reader to Özer and Wei (2001), an online addendum.

Theorem 1. *For any o, the following statements are true for all t:*

1. *$V_t(x, o)$ is convex and coercive in x.*
2. *A state-dependent modified base stock policy is optimal and the optimal base stock level is given by $y_t(o) \equiv \min\{y: V_t(y, o) \leqslant V_t(x, o) \text{ for all } x\}$.*
3. *$J_t(x, o)$ is convex and coercive in x.*

Next, we show that the optimal base stock level is increasing in observed demand beyond the lead time. Specifically, the observed demand beyond the lead time that is closer to the current period has more impact on the optimal base stock level than the observed demand further in the future; that is, an additional unit of advance order to be delivered s periods later increases the optimal base stock level for the current period more than an additional unit of advance order to be delivered $s' > s$ periods later. When capacity is tight and the manufacturer faces one additional unit of observed demand beyond the lead time, one might expect the base stock level to increase more than one unit. We show, however, that an increase of ϵ units in the observed demand does not increase the base stock level for the current period more than ϵ no matter how tightly constrained the capacity is. This line of monotonicity results limit the search for the optimal policy, and reduce the computational requirement for large scale problems. We refer the reader to Porteus (2002) and Veinott (2001) for examples of monotone optima in inventory control and their use.

We define e_j as the $(N-L-1)$-dimensional unit vector whose jth element is 1. Therefore, $o + \epsilon e_j$ adds ϵ units to the observed demand j periods beyond the lead time; that is, $o_{t,t+L+j} + \epsilon$. We also define the first difference of a function $f(x, y)$ as $\nabla f(x, y) \equiv f(x+1, y) - f(x, y)$.

Lemma 1. *Let $f(x)$ and $g(x)$ be two convex and coercive functions, and let x_f and x_g be the smallest minimizer of $f(\cdot)$ and $g(\cdot)$, respectively. If $\nabla f(x) \leqslant \nabla g(x)$ for all x, then $x_f \geqslant x_g$.*

Lemma 2. *For any o and $\epsilon > 0$, we have $\nabla V_t(x-\epsilon, o) \leqslant \nabla V_t(x, o+\epsilon e_1)$, $y_t(o+\epsilon e_1) - y_t(o) \leqslant \epsilon$, and $\nabla J_t(x-\epsilon, o) \leqslant \nabla J_t(x, o+\epsilon e_1)$ for all t.*

Theorem 2. *For any o and $\epsilon > 0$, the following statements are true for all t:*

1. *$y_t(o_2) \geqslant y_t(o_1)$ for any $o_2 \geqslant o_1$.*[4]
2. *$y_t(o+\epsilon e_j) \geqslant y_t(o+\epsilon e_{j+1})$ for $j = 1, \ldots, N-L-2$.*
3. *$y_t(o+\epsilon e_j) - y_t(o) \leqslant \epsilon$ for $j = 1, \ldots, N-L-1$.*

Next, we investigate the sensitivity of the optimal cost function and base stock level with respect to the capacity. At each period, the system runs the risk of not being able to produce enough to satisfy the uncertain demand due to the production capacity. Given observed demand beyond the lead time, one may produce up front and carry additional inventory or one may choose not to produce and instead incur a penalty cost by producing in a future period to reduce the holding cost. Hence, additional capacity can move production in either direction. Veinott (2001) provides examples for both cases. Under a convex single-period cost function, however, we prove that the optimal base stock level is always lower with additional capacity, given the same observed demand beyond the lead time.

Theorem 3. *For any o and $Q_2 > Q_1 > 0$, the following statements are true for all t:*
1. $y_t(o \mid Q_2) \leqslant y_t(o \mid Q_1)$.
2. $J_t(x, o \mid Q_2) \leqslant J_t(x, o \mid Q_1)$.

This result extends the well-known result for the classical case (Federgruen and Zipkin 1986) to account for advance demand information and nonstationary data. Note, however, that different observed demand beyond the lead time may reverse this relationship (see Theorem 2). When the observed demand is large enough, a production system with high capacity may have an optimal base stock level larger than that of a system with low capacity.

Unlike the previous results of this section, Corollary 1 and Theorem 4 below require the assumption of stationary costs and demand distributions, hence, the single-period cost function $G(x)$ is independent of time t. We define $y^m = \min\{y\colon G(y) = \min_x G(x)\}$. We call a base stock policy with base stock level y^m a myopic policy because it ignores the future consequences of the current decision.

Corollary 1. *For a stationary problem, as the capacity $Q \to \infty$, $y_t(o \mid Q) \to y^m$.*

Because $y_t(o \mid Q)$ is convergent, for any $\epsilon > 0$, there exists a $\bar{Q}$ such that for any Q_1 and Q_2 greater than $\bar{Q}$, $|y_t(o \mid Q_1) - y_t(o \mid Q_2)| \leqslant \epsilon$. This fact suggests a capacity above which the reduction in base stock level from additional capacity is negligible and, hence, the cost saving. Additional flexibility in the production system offers a diminishing return. Another way to interpret this result is as follows. The manufacturer may optimally decompose the production problem into two independent problems: produce for observed part of the demand and produce to hedge against the unobserved part of the demand. One can even ignore observed demand information beyond the production lead time.

Consider a problem with *stationary* data, *zero* production lead time $L = 0$, *infinite* capacity, and information horizon $N = 2$. At the beginning of each period t the manufacturer would know $(O_{t,t}, O_{t,t+1})$. For this case it would be optimal not to consider the information $O_{t,t+1}$ for current period production and separate the problem into two parts: produce $O_{t,t}$ and produce to account for the uncertain part $U_{t,t}$ given that there is infinite capacity and the costs are stationary. Now consider the same example with *nonstationary* data and *limited* capacity. In this case, ignoring $O_{t,t+1}$ and determining the optimal production quantity with the same decomposition would be suboptimal. For instance, if the production cost in the following period is higher than the current period, or if $U_{t,t}$ (which is uncertain) has high probability of a low realization and $O_{t,t+1}$ (which is certain) is large, then in this case the manufacturer does not need to hedge against the uncertainty in the current period, but she faces a high probability of capacity shortage in the following period. The optimal policy could use the remaining capacity in the current period to produce and prepare for the following periods.

Next, we investigate the intertemporal relationship of the cost function and the optimal base stock level. We show that the optimal base stock level decreases as one approaches the end of the planning horizon. This behavior suggests that with fewer remaining periods to termination, the manufacturer needs less inventory to cover the potential high demand and shortage in capacity. As time gets closer to the terminal period, the manufacturer has fewer periods to carry inventory to smooth out the fluctuations in demand and compensate for capacity shortages.

Theorem 4. *For any o, the following statements are true for all t:*
1. $\nabla V_{t-1}(x, o) \leqslant \nabla V_t(x, o)$, $x \leqslant y_t(o)$.
2. $y_{t-1}(o) \geqslant y_t(o)$.
3. $J_t(x, o) \geqslant J_{t+1}(x, o)$.

These results are not true in general. In the absence of advance demand information, Chan and Muckstadt (1999) provide a counterexample to Part 2 of Theorem 4 for a production order that is constrained both from below and above. Such a production facility is said to have smoothing constraints.

4. Analysis for Positive Fixed Costs

When the fixed cost is large relative to inventory related costs, or when capacity is low, one may consider producing full capacity only, that is, either producing full capacity Q or not producing at all. We find examples of this policy in process industries with continuous production such as oil and gas, and sugar refining. The high cost of equipment to maintain continuous production requires that a large volume of production is maintained to recover the cost of specialized equipment (Chapter 6 in Schroder 1993). Also, transportation requirements often constrain manufacturers to "full truck load." Within this class of policies, we show that a state-dependent threshold separates the produce full capacity region from the produce nothing region. The Bellman equation is given by

$$J_t(x, o) = \min\{K_t + V_t(x + Q, o), V_t(x, o)\}$$
$$= V_t(x, o) + \min\{H_t(x, o), 0\}, \tag{5}$$

$$H_t(x, o) = K_t + V_t(x + Q, o) - V_t(x, o), \tag{6}$$

$$s_t(o) \equiv \min\{x\colon H_t(x, o) \geqslant 0\}, \tag{7}$$

where $V_t(x, o)$ is defined in Equation (4) and $J_{T+1} \equiv 0$. If the set $\{x\colon H_t(x, o) \geqslant 0\} = \varnothing$, then we define $s_t(o) = \infty$.

The following theorem establishes the optimality of a state-dependent threshold policy.

THEOREM 5. *For any o and for all t:*
1. *$H_t(x, o)$ is increasing in x.*
2. *An optimal policy is given by the threshold $s_t(o)$: If the modified inventory position $x < s_t(o)$, then it is optimal to produce full capacity Q and not to produce otherwise.*

Next, we show that the threshold $s_t(o)$ is increasing in the observed demand beyond the lead time, which parallels the result for the zero fixed cost case. If the modified inventory position x_t is low, an additional unit of observed demand beyond the lead time may trigger production of Q units in the current period. Likewise, the observed demand beyond the lead time that is closer to the current period has more impact on the optimal threshold. The increase in the optimal threshold is always less than the increase in the observed demand beyond the lead time.

LEMMA 3. *For any o and $x_1 \leqslant x_2$, we have $V_t(x_1+Q, o) - V_t(x_1, o) \leqslant V_t(x_2+Q, o) - V_t(x_2, o)$ and $J_t(x_1+Q, o) - J_t(x_1, o) \leqslant J_t(x_2+Q, o) - J_t(x_2, o)$.*

LEMMA 4. *For any o and $\epsilon > 0$, we have $H_t(x-\epsilon, o) \leqslant H_t(x, o+\epsilon e_1)$ and $s_t(o+\epsilon e_1) - s_t(o) \leqslant \epsilon$ and $J_t(x-\epsilon+Q, o) - J_t(x-\epsilon, o) \leqslant J_t(x+Q, o+\epsilon e_1) - J_t(x, o+\epsilon e_1)$ for all t.*

THEOREM 6. *For any o and $\epsilon > 0$, the following statements are true for all t:*
1. *$s_t(o_2) \geqslant s_t(o_1)$ for any $o_2 \geqslant o_1$.*
2. *$s_t(o+\epsilon e_j) \geqslant s_t(o+\epsilon e_{j+1})$ for $j = 1, \ldots, N-L-2$.*
3. *$s_t(o+\epsilon e_j) - s_t(o) \leqslant \epsilon$ for $j = 1, \ldots, N-L-1$.*

Recall that in the previous section for the zero fixed cost case, we proved that both the optimal base stock level and the optimal cost are decreasing in capacity (Theorem 3). When the fixed cost is positive, however, neither the optimal threshold nor the optimal cost is necessarily decreasing in the capacity. In this case, depending on the current and future costs, increasing capacity can either increase or decrease the threshold and the cost. The reason for this seemingly counterintuitive result is because the policy is to either produce nothing or use the full capacity. For the optimal cost, a higher capacity may impose a higher inventory level and, hence, a higher holding cost without reducing the shortage cost much. For the optimal threshold, when the capacity is tight, producing full capacity may not be enough to reduce future shortage costs. So increasing capacity would increase the threshold (to cope with future shortages) and make it more likely to produce full capacity. However, when capacity is large enough, increasing capacity would increase the cost of producing at full capacity and carrying additional inventory. The savings from pooling production orders and reducing shortage costs may not be high enough to offset the production and holding costs. In this case, the threshold decreases with an increase in capacity, making it less likely to produce full capacity. Later in the numerical study we provide an example for this non-monotonic behavior.

Next, we address the sensitivity of the optimal threshold and the cost with respect to the fixed cost K_t. Let K^i denote the vector of fixed costs for periods t through T, that is, $K^i = (K_t^i, K_{t+1}^i, \ldots, K_T^i)$. We say $K^2 \geqslant K^1$ if for all t, $K_t^2 \geqslant K_t^1$. One might expect the optimal cost to increase with K_t. Similarly, a larger fixed cost could make it less likely to produce full capacity and decrease the threshold. We show, however, that this monotonic behavior is true only under a stronger condition.

THEOREM 7. *For any o:*
1. *If $K^2 > K^1 > 0$, then $J_t(x, o \mid K^2) \geqslant J_t(x, o \mid K^1)$.*
Furthermore, if $K_t^2 - \alpha K_{t+1}^2 \geqslant K_t^1 - \alpha K_{t+1}^1$ for all t, then
2. *$H_t(x, o \mid K^2) \geqslant H_t(x, o \mid K^1)$, and*
3. *$s_t(o \mid K^2) \leqslant s_t(o \mid K^1)$.*

Consider a problem with fixed costs K^1. We now increase the fixed cost at period $t+1$ by a large quantity and refer to this new fixed cost structure as K^2. Intuitively, this increase in the fixed cost would make it less likely to produce in period $t+1$ and more likely to produce in period t. Hence, we would expect $s_t(o \mid K^1) \leqslant s_t(o \mid K^2)$. The condition in Theorem 7 specifies exactly how large K_{t+1} can be for this monotonicity result to hold. Note also that the stationary fixed cost problem $K_t = K$ is a special case.

Next, under stationary costs and demand distributions, we show that the optimal threshold $s_t(o)$ and the cost function $J_t(x, o)$ are decreasing in t. As we get closer to the terminal period the system becomes less responsive to a production order of full capacity. The manufacturer carries less inventory by reducing the threshold levels. We also use this result for the proof of the infinite horizon problem.

THEOREM 8. *For any o, the following statements are true for all t:*
1. *$H_{t-1}(x, o) \leqslant H_t(x, o)$ for $x \leqslant s_t(o)$.*
2. *$s_{t-1}(o) \geqslant s_t(o)$.*
3. *$J_t(x, o) \geqslant J_{t+1}(x, o)$.*

5. Infinite Horizon Problem

In this section, we show that a state-dependent modified base stock policy and a state-dependent threshold policy are also optimal for the infinite horizon problem. First, we present the necessary assumptions.

ASSUMPTION 1. *$G(y) = O(|y|^p)$ for some positive integer p.*[5]

ASSUMPTION 2. *$E(D_{t,s})^q < \infty$ for all $q \leqslant p$.*

Recall that $J_t(x, o)$ is the optimal expected cost for the $(T-t)$-period problem. To emphasize its dependence on the final period T, we denote the optimal cost function as $J_t(x, o \mid T)$. The demand distribution and the cost parameters are stationary. Hence, the lead time demand

$\sum_{s=t}^{t+L+1} D_{t,s}$ is stationary over time and we refer to it as Z. The following theorems consider the limit of function $J_t(x, o \mid T)$ and $V_t(x, o \mid T)$ as $T \to \infty$ for both zero and positive fixed cost problems.

Theorem 9. *For the zero fixed cost case and any vector o:*

1. *The sequence $\{J_t(x, o \mid T)\}$ converges pointwise to a limit $J(x, o)$ that is convex and coercive.*
2. *The function $V(x, o) = G(x) + \alpha EJ(x - o^1 - Z, O)$, where o^1 is the first element of vector 0, is well defined, and V is convex and has at least one finite minimizer. Let $y_\infty(o)$ be its smallest minimizer. Then, we have $\lim_{T\to\infty} y_t(o \mid T) = y_\infty(o)$.*
3. *The optimal cost function J satisfies the Bellman equation $J(x, o) = \min_{x\leqslant y\leqslant x+Q}\{G(y) + \alpha EJ(y - o^1 - Z, O)\}$ and the minimum is achieved by $y_\infty(o)$. Hence, a state-dependent modified base stock policy is optimal.*

Theorem 10. *For the positive fixed cost case and any vector o:*

1. *The sequence $\{J_t(x, o \mid T)\}$ converges pointwise to a limit $J(x, o)$.*
2. *The function $V(x, o) = G(x) + \alpha EJ(x - o^1 - Z, O)$ and $H(x, o) = K + V(x+Q, o) - V(x, o)$ are well defined; $H(x, o)$ is increasing. Let $s_\infty(o) = \min\{x: H(x, o) \geqslant 0\}$. Then, we have $\lim_{T\to\infty} s_t(o \mid T) \to s_\infty(o)$.*
3. *The optimal cost function J satisfies the Bellman equation $J(x, o) = \min\{K + V(x+Q, o), V(x, o)\}$. Hence, the optimal policy is given by the state-dependent threshold $s_\infty(o)$.*

6. Numerical Study

We carry out a numerical study (1) to verify some of our structural results and enhance our understanding of the problem, (2) to provide a counterexample and show the nonexistence of a structural property, and (3) to illustrate how one can use the present model and quantify the system's performance with respect to factors such as the advance demand information and the capacity. For certain problem instances, we address the following: the impact of advance demand information and capacity on the optimal policy and the cost of managing a production system; the trade-off between investing in capacity and obtaining advance demand information; and the impact of holding inventory versus incurring shortage cost.

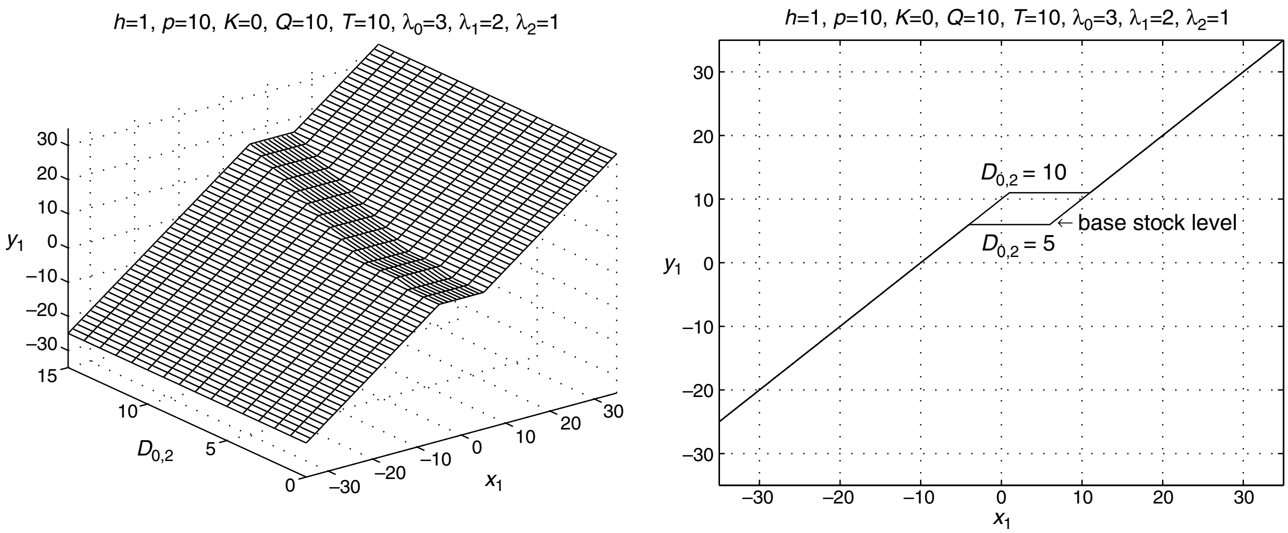
We use a backward induction algorithm to solve the functional Equation (3). We do not use any of our structural results to enhance this well-known algorithm. By doing so we verify our structural results. Recall that the state space of our problem is of dimension $1 + (N - L - 1)^+$. In this section, we limit ourselves to two-dimensional problems by setting $N = L + 2$. The computational requirement for a backward induction algorithm increases with the dimension of the state space. Note that the vector $O_t = (O_{t,t+L+1}) = D_{t-1,t+1}$ is a scalar in this case. We study 350 problem instances, all with linear holding and penalty costs and zero lead time. In addition, all cost parameters are stationary, that is, $h_t = h$, $p_t = p$, $c_t = c$, and $K_t = K$. We fix $h = 1$, $c = 2$, $\alpha = 0.9$, and $N = 2$ and change the penalty cost, fixed cost, capacity, and advance demand information scenario. We use $K = 0, 10, 20, 50, 100$, $p = 1, 5, 10, 50$, and $Q = 3, 4, \ldots, 12$.

The demand vector for $N = 2$ is given by $(D_{t,t}, D_{t,t+1}, D_{t,t+2})$. We model $D_{t,t+i}$ as a Poisson distribution with mean λ_i. Recall that demand at any period s is given by $D_{s-2,s} + D_{s-1,s} + D_{s,s}$, with mean $\lambda_2 + \lambda_1 + \lambda_0$. Note that during period $s-2$, $D_{s-2,s}$ will be revealed. In other words, on average the production manager will observe $[\lambda_i/(\lambda_2 + \lambda_1 + \lambda_0)] \cdot 100\%$ of the total demand for period t in period $t - i$, that is, i periods in advance. Consider, for example, the extreme case in which the manager implements an aggressive discount strategy and convinces all her customers to place their orders two periods in advance. This scenario could be modeled by setting $\lambda_0 = \lambda_1 \equiv 0$ and $\lambda_2 > 0$.

First, we address systems with zero fixed costs. In Figure 2(a), we plot y_t as a function of x_t and $D_{t-1,t+1}$. We observe the optimality of a state-dependent modified base stock policy; that is, if $x_t \leqslant y_t(D_{t-1,t+1})$, then produce up to the minimum of $y_t(D_{t-1,t+1}) - x_t$ and the capacity. Otherwise, do nothing (Theorem 1). We also observe that the optimal base stock level is increasing in $D_{t-1,t+1}$ (Theorem 2). Figure 2(b) is the cross-section of (a) at $D_{t-1,t+1} = 5, 10$.

In Figures 3(a) and 3(b), we plot the *optimal* expected cost and the *optimal* base stock levels with respect to the advance demand information scenario and capacity. The top curves in both figures correspond to the scenario with no advance demand information; whereas, the bottom curves correspond to the scenario in which the manager convinces all her customers to place orders two periods in advance. We observe that both the cost and the base stock level decrease as customers place more of their demand in advance. For example, if 50% of the demand is revealed one period in advance for a system with four units of capacity, the cost decreases by 26% (moving from a capacitated system A to system B in Figure 3(a)). Next, we observe that both the cost and the base stock level decrease with an increase in capacity (Theorem 3), but increasing capacity offers diminishing returns. The cost reduction is 40% from system A to system C; whereas, it is 15% from system D to system E. This observation suggests an optimal capacity expansion schedule depending on its cost. These figures also suggest that advance demand information is even more valuable for systems with tight capacity. Finally, we observe that advance demand information is a substitute both for capacity and inventory. The comparison of system D and system F exemplifies this fact. The two systems have similar costs. System D has one more unit of capacity than system F, whereas 50% of the customers in system F reveal their requirements one period earlier than they do in

Figure 2. $K = 0$.



(a) Structure of the optimal policy.

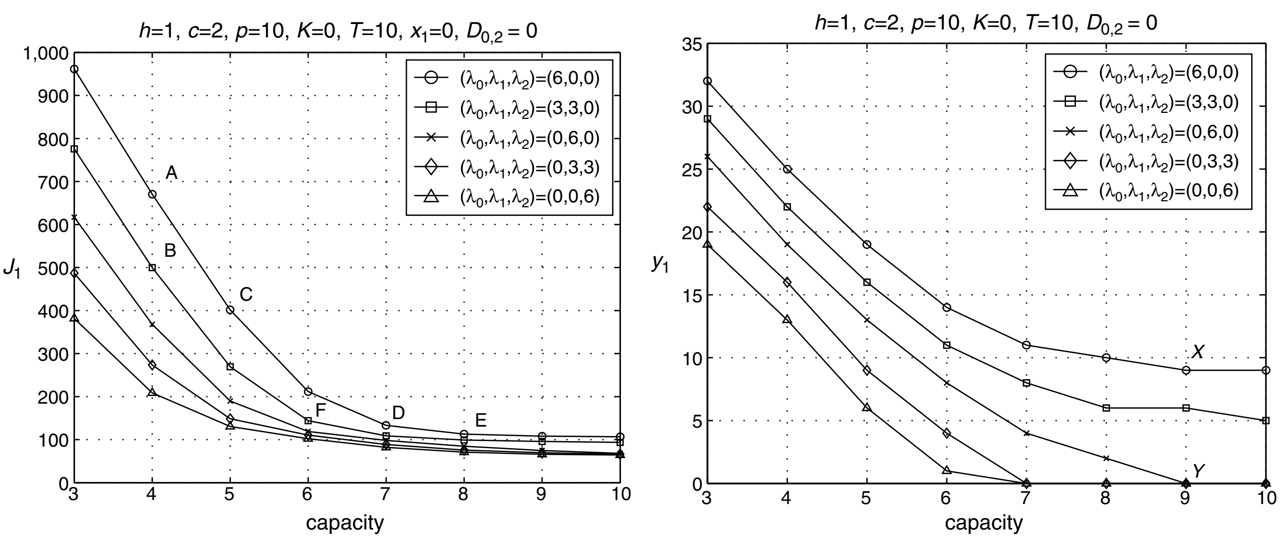
(b) Cross-section of (a) at $D_{0,2} = 5, 10$.

system D. Similarly, system Y in Figure 3(b) maintains a zero base stock level due to advance demand information.

In Table 1, we report the cost and optimal base stock levels for additional problem instances with varying penalty cost ($p = 1, 10, 50$), planning horizons ($T = 10, 20$), and capacity ($Q = 3, 6, 9, 12$). In this table, we quantify the observations made in the previous paragraph and report the percentage cost reduction due to advance demand information. For example, comparing experiment number 1 with experiment number 5 reveals that the expected cost decreases by 29.7% ($=(143.9 - 101.2)/143.9$) and the base stock level drops to zero if all the customers place their orders two periods in advance. For high penalty cost or low capacity systems, we observe that the reduction in inventory costs due to advance demand information is even larger. For example, the reduction in cost is 29.7% from experiment number 1 to experiment number 5, while it is 64.7% from experiment number 11 to experiment number 15. We also observe that the cost and base stock level decrease convexly with advance demand information, suggesting that convincing customers to place advance orders has diminishing returns. We also observe the joint role of capacity expansion and advance demand information.

Next, we address systems with a positive fixed cost. In Figure 4(a), we observe the optimality of a state-dependent threshold policy for the fixed cost case, that

Figure 3. $K = 0$.



(a) Cost vs. advance demand information and capacity.

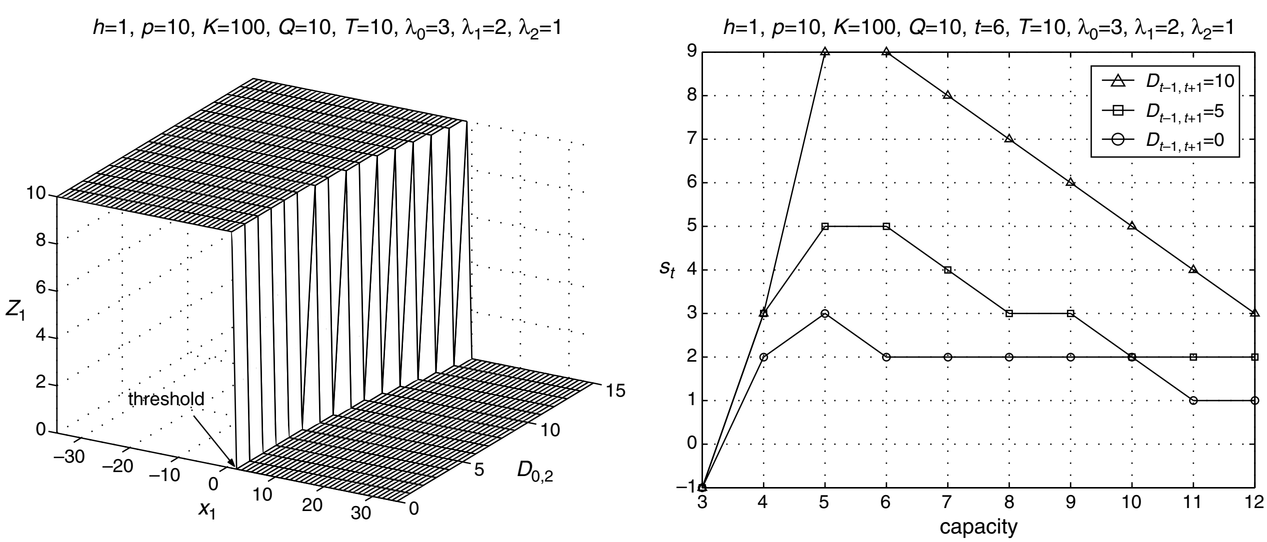
(b) Base stock level vs. advance demand information and capacity.

Table 1. Optimal base stock level and cost under varying capacity and advance demand information when $h = 1$, $c = 2$, and $K = 0$.

Exp. no.	Capacity $(\lambda_0, \lambda_1, \lambda_2)$	$y_1(0)$ 3	6	9	12	Cost: $J_1(0,0)$ 3	6	9	12	Cost reduction (%) 3	6	9	12
1	(6, 0, 0)	13	7	5	5	**143.9**	89.8	83.5	83.1				
2	(3, 3, 0)	10	4	3	2	126.1	83.1	76.3	75.7	12.4	7.4	8.7	8.9
3	(0, 6, 0)	7	0	0	0	113.2	75.1	64.2	63.4	21.4	16.3	23.1	23.7
4	(0, 3, 3)	4	0	0	0	105.8	72.2	64.1	63.4	26.5	19.6	23.3	23.7
5	(0, 0, 6)	0	0	0	0	**101.2**	70.8	64.0	63.4	**29.7**	21.1	23.3	23.7
$p = 1$, 10 periods													
6	(6, 0, 0)	32	14	9	9	961.4	211.7	107.9	105.9				
7	(3, 3, 0)	29	11	6	5	775.7	144.1	95.6	92.7	19.3	31.9	11.4	12.6
8	(0, 6, 0)	26	8	0	0	616.8	118.9	74.6	64.1	35.8	43.8	30.9	39.5
9	(0, 3, 3)	22	4	0	0	487.1	110.3	69.4	63.9	49.3	47.9	35.7	39.7
10	(0, 0, 6)	19	1	0	0	381.7	102.0	65.9	63.4	60.3	51.8	39.0	40.2
$p = 10$, 10 periods													
11	(6, 0, 0)	38	20	12	11	4,594	730	131	121				
12	(3, 3, 0)	38	16	7	7	3,663	359	110	104	20.3	50.8	16.3	13.7
13	(0, 6, 0)	35	13	3	0	2,855	202	91	67	37.9	72.4	30.8	44.3
14	(0, 3, 3)	31	9	0	0	2,181	147	78	65	52.5	79.8	40.7	46.1
15	(0, 0, 6)	28	6	0	0	1,619	130	70	63	**64.7**	82.2	46.4	**47.6**
$p = 50$, 10 periods													
16	(6, 0, 0)	49	16	9	9	1,964	328.0	145.9	142.9				
17	(3, 3, 0)	46	13	6	5	1,710	233.5	129.3	125.0	12.9	28.8	11.4	12.6
18	(0, 6, 0)	43	9	0	0	1,483	192.0	101.7	86.5	24.5	41.5	30.3	39.5
19	(0, 3, 3)	40	6	0	0	1,287	178.0	94.7	86.3	34.5	45.7	35.0	39.7
20	(0, 0, 6)	37	3	0	0	1,107	169.1	89.5	85.5	43.6	48.4	38.6	42.2
$p = 10$, 20 periods													

is, if $x_t \leqslant s_t(D_{t-1,\,t+1})$, then it is optimal to produce full capacity $z_t = Q = 10$. Note that the threshold increases as $D_{t-1,\,t+1}$ exceeds five units (Theorem 6). If there is more demand to come in the future, one can produce more today, incur the fixed cost, and carry this additional inventory to avoid shortage and penalty cost in the future. Unlike the zero fixed cost case, increasing capacity may increase both the threshold and the optimal cost in positive fixed cost

Figure 4. $K > 0$.



(a) Structure of the optimal policy.

(b) Threshold vs advance demand information and capacity.

Table 2. Optimal threshold and the associated cost when $h=1$, $p=10$, $c=2$, and $T=10$ periods.

	Capacity	3	6	9	12	3	6	9	12	3	6	9	12
No.	$(\lambda_0, \lambda_1, \lambda_2)$	$s_1(0)$				Cost: $J_1(0,0)$				Cost reduction (%)			
1	(6, 0, 0)	28	11	7	6	**1,027.0**	275.9	166.4	162.8				
2	(3, 3, 0)	25	8	3	2	**840.9**	207.2	154.6	153.3	**18.1**	24.9	7.1	5.9
3	(0, 6, 0)	22	5	0	−1	681.9	180.5	132.1	129.7	33.6	34.6	20.6	20.4
4	(0, 3, 3)	19	1	−1	−1	552.2	169.9	129.4	129.2	46.2	38.4	22.3	20.6
5	(0, 0, 6)	16	0	−1	−1	446.7	154.6	119.8	118.9	56.5	44.0	28.0	27.0
$K=10$													
6	(6, 0, 0)	20	10	6	5	**1,279.0**	526.7	351.9	305.8				
7	(3, 3, 0)	17	6	2	2	1,093.0	450.3	327.3	286.6	14.5	14.5	7.0	6.3
8	(0, 6, 0)	14	3	−1	−1	934.5	410.6	287.3	250.5	26.9	22.0	18.4	18.1
9	(0, 3, 3)	11	0	−1	−1	804.7	377.7	273.2	241.3	37.1	28.3	22.4	21.1
10	(0, 0, 6)	8	0	−1	−1	698.6	343.3	249.6	220.0	45.4	34.8	29.1	28.1
$K=50$													

problems. To illustrate this behavior, we provide examples for the threshold in Figure 4(b) and examples for the optimal cost in Table 3 (see also the intuitive explanation provided after Theorem 6).

In Table 2, we illustrate the gains from reducing fixed costs and how they compare to those of advance demand information. While advance demand information is more valuable when the capacity is tight, the fixed cost reduction is more effective when the capacity is abundant. For example, when $Q=3$, convincing 50% of customers to place orders one period in advance reduces optimal cost by 18.1%. To achieve a similar cost reduction of 19.7% (=(1,279 − 1,027)/1,279), the manufacturer must reduce the fixed cost from 50 to 10 (see experiment numbers 1, 2, and 6). On the other hand, when $Q \geqslant 6$, the same reduction in the fixed cost is more effective than obtaining advance demand information. In Table 3, we illustrate the sensitivity of the system with respect to the fixed cost. For example, reducing the fixed cost from $K=100$ to 5 increases the threshold from 9 to 25, increasing the order frequency while reducing the cost by 42.6% (=(1,319 − 756.4)/1,319) (Theorem 7).

One of the major decisions in a capacity problem is the expansion (contraction) size. The expansion cost $C(\Delta Q)$ may take several forms; such as linear, power or step cost function (Luss 1982). If capacity expansion is a one time decision, then the manufacturer's problem is to solve $\min_{\Delta Q}\{C(\Delta Q) + J_t(x, O \mid Q_0 + \Delta Q)\}$. In Figure 5 we provide an example of this problem when $C(\Delta Q) = 100 * \Delta Q$ and $Q_0 = 3$ under two advance demand information scenarios. For this particular example, convincing customers to place orders two periods in advance reduces the *optimal* capacity expansion decision from $Q_1^* - Q_0 = 3$ to $Q_2^* - Q_0 = 1$. This is another example illustrating how advance demand information can be a substitute for capacity. Note that this approach for capacity expansion decision can be used in a rolling horizon fashion; that is, one can change the capacity of the production system whenever the authority is given to do so.

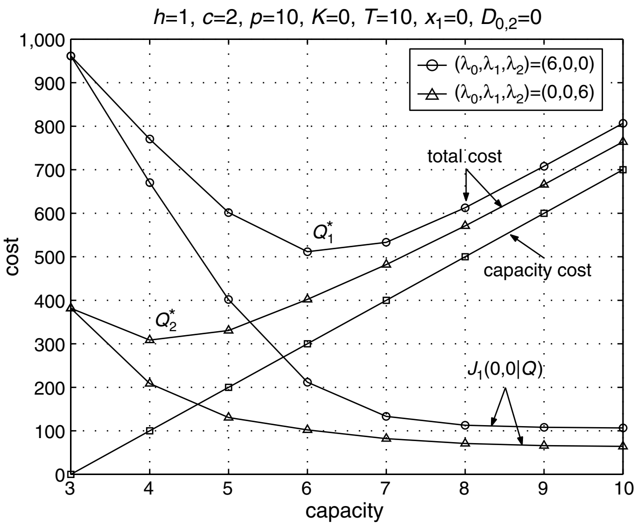
7. Conclusion

In this paper, we establish optimal policies for a capacitated inventory system with advance demand information. For the zero fixed cost case, we establish the optimality of a state-dependent modified base stock policy and analyze the properties of the optimal policy. For the positive fixed cost case, we analyze a class of production policies under which the manager is restricted to either producing at full capacity or not at all. Within this class of policy, we prove that a state-dependent threshold characterizes an optimal replenishment strategy. In addition to verifying the analytical results, our numerical study illustrates how advance demand information mitigates the shortfall of limited production capacity. In this sense advance demand informa-

Table 3. Sensitivity with respect to the positive fixed cost when $(\lambda_0, \lambda_1, \lambda_2) = (3, 2, 1)$ and $T=10$ periods.

	Capacity	3	6	9	10	11	12	3	6	9	10	11	12	
Exp. no.	K	$s_1(0)$						Cost: $J_1(0,0)$						
1	5	**25**	7	3	3	3	2	**756.4**	165.5	**131.6**	**132.1**	**133.8**	**135.9**	←
2	10	24	7	3	3	2	2	788.9	196.1	153.2	**151.8**	**151.9**	**152.6**	←
3	20	22	7	3	2	2	2	854.1	256.9	195.9	190.8	187.5	185.6	
4	50	16	5	2	2	2	2	1,042.0	435.9	321.6	304.7	292.3	283.0	
5	100	**9**	4	2	2	1	1	**1,319.0**	715.3	521.7	487.9	461.3	439.7	

Figure 5. Optimal capacity vs. advance demand information.



tion can be a substitute for capacity and inventory. We also illustrate the increasing value of advance demand information for inventory systems with tight capacity. Supply chain managers and decision makers are often interested in understanding qualitatively how they should respond to the changes in the environment of the problem. The structural results in this paper provide ways for developing insights as well as tools for designing efficient algorithms for large scale problems. The model with a numerical study provides a framework to quantify the performance of a capacitated system with respect to advance demand information, capacity, and cost parameters.

Appendix. Proofs

Proof of Corollary 1. Gallego and Özer (2001) show that a myopic policy with base stock level y^m is optimal for an uncapacitated stationary system. Part 1 of Theorem 3 shows that $y_t(o \mid Q)$ is a decreasing sequence in Q. Therefore, $y_t(o \mid Q)$ monotonically converges to y^m. □

Proof of Theorem 1. Part 1 holds for $t = T$ because $V_T(x, o) = G_T(x)$. Assume it holds for t. Then, there exists a finite minimizer of function $V_t(\cdot, o)$. This implies Part 2 for t. Under this policy, the modified inventory position after production order is $y_t = x$ if $y_t(o) \leqslant x$; $y_t = y_t(o)$ if $y_t(o) - Q \leqslant x < y_t(o)$; and $y_t = x + Q$ if $x < y_t(o) - Q$. The optimal cost function at t is given by

$$J_t(x, o) = \begin{cases} V_t(x+Q, o), & x < y_t(o) - Q, \\ V_t(y_t(o), o), & y_t(o) - Q \leqslant x < y_t(o), \\ V_t(x, o), & y_t(o) \leqslant x. \end{cases} \quad (8)$$

This function is convex and coercive in x (Part 3 for t). We now prove Part 1 for $t-1$. From Equation (4), $V_{t-1}(x, o)$ is convex and coercive because $G_t(x)$ is convex and coercive and the update for the modified inventory position is linear and expectation preserves convexity and coerciveness. This concludes the induction argument and the proof. □

Proof of Theorem 2. For Part 1, we prove $\nabla V_t(x, o_2) \leqslant \nabla V_t(x, o_1)$. This holds as an equality for $t = T$. Assume for an induction that it holds for t. This implies that $y_t(o_2) \geqslant y_t(o_1)$ due to Lemma 1. Next, we show that this implies $\nabla J_t(x, o_2) \leqslant \nabla J_t(x, o_1)$ for t. To do so we use Equation (8), the induction argument, and consider three cases:

Case 1. If $x < y_t(o_1) - Q$, then $\nabla J_t(x, o_1) = \nabla V_t(x+Q, o_1) \geqslant \nabla V_t(x+Q, o_2) = \nabla J_t(x, o_2)$.

Case 2. If $y_t(o_1) - Q \leqslant x < y_t(o_2)$, then $\nabla J_t(x, o_1) \geqslant 0 \geqslant \nabla J_t(x, o_2)$.

Case 3. If $y_t(o_2) \leqslant x$, then $\nabla J_t(x, o_1) = \nabla V_t(x, o_1) \geqslant \nabla V_t(x, o_2) = \nabla J_t(x, o_2)$.

To conclude the induction argument, we show that for $t-1$, $\nabla V_{t-1}(x, o_2) = \nabla G_{t-1}(x) + \alpha E \nabla J_t(x_2, O_2) \leqslant \nabla G_{t-1}(x) + \alpha E \nabla J_t(x_1, O_2) \leqslant \nabla G_{t-1}(x) + \alpha E \nabla J_t(x_1, O_1) = \nabla V_{t-1}(x_1, o_1)$, where x_i refers to the next period's modified inventory position when the current period state space is (x, o_i) for $i = 1, 2$. The first inequality is due to the convexity of $J_t(x, o)$ and that $x_2 \leqslant x_1$ (due to the state update (1) and $o_1 < o_2$). Similarly, the second inequality is from the induction argument, the state update (2) (which implies $O_1 \leqslant O_2$ together with $o_1 < o_2$), and the fact that expectation preserves inequality.

For Part 2, note that the third part of Lemma 2 implies that $\nabla V_t(x, o + \epsilon e_1) = \nabla G_t(x) + \alpha E \nabla J_{t+1}(x - o_{t,t+L+1} - \epsilon - \sum_{s=t}^{t+L+1} D_{t,s}, O_{t+1}) \leqslant \nabla G_t(x) + \alpha E \nabla J_{t+1}(x - o_{t,t+L+1} - \sum_{s=t}^{t+L+1} D_{t,s}, O_{t+1} + \epsilon e_1) = \nabla V_t(x, o + \epsilon e_2)$ for all t. Hence, for all t,

$$\nabla V_t(x, o + \epsilon e_j) \leqslant \nabla V_t(x, o + \epsilon e_{j+1}) \quad (9)$$

is true for $j = 1$. Now assume for an induction argument that Equation (9) is true for j, then Part 2 for j follows from Lemma 1. Next, we show that Part 2 for j and the induction argument imply $\nabla J_t(x, o + \epsilon e_j) \leqslant \nabla J_t(x, o + \epsilon e_{j+1})$. To do so, we use Equation (8) and consider three cases:

Case 1. If $x < y_t(o + \epsilon e_{j+1}) - Q$, then $\nabla J_t(x, o + \epsilon e_{j+1}) = \nabla V_t(x+Q, o + \epsilon e_{j+1}) \geqslant \nabla V_t(x+Q, o + \epsilon e_j) = \nabla J_t(x, o + \epsilon e_j)$.

Case 2. If $y_t(o + \epsilon e_{j+1}) - Q \leqslant x < y_t(o + \epsilon e_j)$, then $\nabla J_t(x, o + \epsilon e_{j+1}) \geqslant 0 \geqslant \nabla J_t(x, o + \epsilon e_j)$.

Case 3. If $y_t(o + \epsilon e_j) \leqslant x$, then $\nabla J_t(x, o + \epsilon e_{j+1}) = \nabla V_t(x, o + \epsilon e_{j+1}) \geqslant \nabla V_t(x, o + \epsilon e_j) = \nabla J_t(x, o + \epsilon e_j)$.

From Equation (4) and the three cases above, $\nabla V_t(x, o + \epsilon e_{j+1}) = \nabla G_t(x) + E \nabla J_{t+1}(x - o_{t,t+L+1} - \sum_{s=t}^{t+L+1} D_{t,s}, O_{t+1} + \epsilon e_j) \leqslant \nabla G_t(x) + E \nabla J_{t+1}(x - o_{t,t+L+1} - \sum_{s=t}^{t+L+1} D_{t,s}, O_{t+1} + \epsilon e_{j+1}) = \nabla V_t(x, o + \epsilon e_{j+2})$. This completes the induction argument and the proof of Equation (9) and Part 2. Part 3 for $j > 1$ follows immediately from Part 2 and Lemma 2 because $y_t(o + \epsilon e_j) - y_t(o) \leqslant y_t(o + \epsilon e_{j-1}) - y_t(o) \leqslant \cdots \leqslant y_t(o + \epsilon e_1) - y_t(o) \leqslant \epsilon$. □

Proof of Theorem 4. For Parts 1 and 2, we prove that for any given o, $\nabla J_t(x, o) \leqslant \nabla J_{t+1}(x, o)$ for $x \leqslant y_t(o)$ by induction. From Equation (8), we have $\nabla J_T(x, o) \leqslant 0 = \nabla J_{T+1}(x, o)$ for $x \leqslant y_T(o)$. Assume for an induction argument that the inequality holds for t. We show that Part 1 is true for t, that is, for $x \leqslant y_t(o)$,

$$\begin{aligned}\nabla V_{t-1}(x, o) &= \nabla G(x) + \alpha E \nabla J_t\left(x - o_{t-1,\,t+L} - \sum_{s=t-1}^{t+L} D_{t-1,\,s}, O\right)\\ &\leqslant \nabla G(x) + \alpha E \nabla J_{t+1}\left(x - o_{t-1,\,t+L} - \sum_{s=t}^{t+L+1} D_{t,\,s}, O\right)\\ &= \nabla V_t(x, o).\end{aligned}$$

The equalities are from the definition of V_t (Equation (4)). The inequality is due to two reasons: stationarity assumption and the induction argument. Under the stationarity assumption the demand vector D_t is i.i.d. Hence, $\sum_{s=t-1}^{t+L} D_{t-1,\,s}$ and $\sum_{s=t}^{t+L+1} D_{t,\,s}$ have the same distribution. Let us refer to them as Z. We need to show that $E\nabla J_t(x - o_{t-1,\,t+L} - Z, O) \leqslant E\nabla J_{t+1}(x - o_{t-1,\,t+L} - Z, O)$. Note that for any o, if $x \leqslant y_t(o) = y_t(o_{t-1,\,t+L}, \ldots, o_{t-1,\,t+N-2})$, then we have

$$\begin{aligned}x - o_{t-1,\,t+L} &\leqslant y_t(0, o_{t-1,\,t+L+1}, \ldots, o_{t-1,\,t+N-2})\\ &\leqslant y_t(o_{t-1,\,t+L+1}, 0, o_{t-1,\,t+L+2}, \ldots, o_{t-1,\,t+N-2})\\ &\leqslant \cdots \leqslant y_t(o_{t-1,\,t+L+1}, \ldots, o_{t-1,\,t+N-2}, 0)\\ &\leqslant y_t(o_{t-1,\,t+L+1} + D_{t-1,\,t+L+1}, \ldots, o_{t-1,\,t+N-2}\\ &\qquad + D_{t-1,\,t+N-2}, D_{t-1,\,t+N-1}) = y_t(O).\end{aligned}$$

The first inequality is due to Part 3 of Theorem 2; the intermediate steps are due to Part 2 of Theorem 2; and the last inequality is due to Part 1 of Theorem 2. Therefore, for any realization of Z and O, $x - o_{t-1,\,t+L} - Z \leqslant y_t(O)$. Now we can use the induction hypothesis to claim that $E\nabla J_t(x - o_{t-1,\,t+L} - Z, O) \leqslant E\nabla J_{t+1}(x - o_{t-1,\,t+L} - Z, O)$ because the expectation would preserve the inequality. Therefore, Part 1 for t holds; we have $\nabla V_{t-1}(x, o) \leqslant \nabla V_t(x, o)$ for $x \leqslant y_t(o)$, and hence $y_{t-1}(o) \geqslant y_t(o)$. We complete the induction argument by considering two cases:

Case 1. $x \leqslant y_t(o) - Q$. Then, from Equation (8) and the induction argument we have $\nabla J_{t-1}(x, o) = \nabla V_{t-1}(x + Q, o) \leqslant \nabla V_t(x + Q, o) = \nabla J_t(x, o)$.

Case 2. $y_t(o) - Q \leqslant x \leqslant y_{t-1}(o)$. Then, from Equation (8) we have $\nabla J_{t-1}(x, o) \leqslant 0 \leqslant \nabla J_t(x, o)$.

The proof for Part 3 is a simple induction. Because $G(y)$ is nonnegative, $J_T(x, o) = \min_{x \leqslant y \leqslant x+Q} G(y) \geqslant 0 = J_{T+1}(x, o)$. Hence, Part 3 is true for $t = T$. Assume it is true for t. Then, $V_{t-1}(x, o) = G(x) + \alpha E J_t(x', O) \geqslant G(x) + \alpha E J_{t+1}(x', O) = V_t(x, o)$. Consequently, $J_{t-1}(x, o) = \min_{x \leqslant y \leqslant x+Q}\{V_{t-1}(y, o)\} \geqslant \min_{x \leqslant y \leqslant x+Q}\{V_t(y, o)\} = J_t(x, o)$. □

Proof of Theorem 5. From Equations (4) and (6), $H_T(x, o) = K_T + G_T(x + Q) - G_T(x)$ and it is increasing in x due to the convexity of $G_T(x)$. Now assume for an induction argument that $H_t(x, o)$ is increasing in x. This implies the existence of $s_t(o)$ (could be ∞) and, hence, Part 2 follows. We have $J_t(x + Q, o) - J_t(x, o) = V_t(x + Q, o) - V_t(x, o) + \min\{H_t(x + Q, o), 0\} - \min\{H_t(x, o), 0\} = -K_t + H_t(x, o) + \min\{H_t(x + Q, o), 0\} - \min\{H_t(x, o), 0\} = -K_t + \max\{H_t(x, o), 0\} + \min\{H_t(x + Q, o), 0\}$. This is the sum of two increasing functions and a constant. Hence, it is increasing. This and the convexity of $G_{t-1}(x)$ imply that $H_{t-1}(x, o) = K_{t-1} + [G_{t-1}(x + Q) + \alpha E J_t(x' + Q, O)] - [G_{t-1}(x) + \alpha E J_t(x', O)]$ is also increasing, completing the induction argument and the proof. Note that under this policy the optimal cost function is

$$J_t(x, o) = \begin{cases} K_t + V_t(x + Q, o), & x < s_t(o),\\ V_t(x, o), & x \geqslant s_t(o). \end{cases} \tag{10}$$

This proof was inspired from Gallego and Toktay (1999). □

Proof of Theorem 6. For Part 1, we prove $H_t(x, o_2) \leqslant H_t(x, o_1)$. This holds as an equality for $t = T$. Assume for an induction argument that it holds for t. This implies $s_t(o_2) \geqslant s_t(o_1)$ because $H_t(x, o)$ is increasing in x (Part 1 of Theorem 5). Next, we prove $J_t(x + Q, o_2) - J_t(x, o_2) \leqslant J_t(x + Q, o_1) - J_t(x, o_1)$ for t by using the induction argument and Equation (10), and considering nine cases:

Case 1. $x < s_t(o_2) - Q$ and $x < s_t(o_1) - Q$. Then, $J_t(x + Q, o_2) - J_t(x, o_2) = H_t(x + Q, o_2) - K_t \leqslant H_t(x + Q, o_1) - K_t = J_t(x + Q, o_1) - J_t(x, o_1)$.

Case 2. $x < s_t(o_2) - Q$ and $s_t(o_1) - Q \leqslant x < s_t(o_1)$. Then, $J_t(x + Q, o_2) - J_t(x, o_2) = H_t(x + Q, o_2) - K_t < -K_t = J_t(x + Q, o_1) - J_t(x, o_1)$ because $H_t(\cdot, o_2) \leqslant 0$ in this domain.

Case 3. $x < s_t(o_2) - Q$ and $s_t(o_1) \leqslant x$. Then,

$$\begin{aligned}J_t(x + Q, o_2) - J_t(x, o_2) &= \underbrace{H_t(x + Q, o_2)}_{<0} - K_t < \underbrace{H_t(x, o_1)}_{\geqslant 0} - K_t\\ &= J_t(x + Q, o_1) - J_t(x, o_1).\end{aligned}$$

Case 4. $s_t(o_2) - Q \leqslant x < s_t(o_2)$ and $x < s_t(o_1) - Q$ is not possible because $s_t(o_1) \leqslant s_t(o_2)$.

Case 5. $s_t(o_2) - Q \leqslant x < s_t(o_2)$ and $s_t(o_1) - Q \leqslant x < s_t(o_1)$. Then, $J_t(x + Q, o_2) - J_t(x, o_2) = -K_t = J_t(x + Q, o_1) - J_t(x, o_1)$.

Case 6. $s_t(o_2) - Q \leqslant x < s_t(o_2)$ and $s_t(o_1) \leqslant x$. Then, $J_t(x + Q, o_2) - J_t(x, o_2) = -K_t \leqslant -K_t + H_t(x, o_1) = J_t(x + Q, o_1) - J_t(x, o_1)$.

Case 7. $s_t(o_2) \leqslant x$ and $x < s_t(o_1) - Q$ is not possible because $s_t(o_1) \leqslant s_t(o_2)$.

Case 8. $s_t(o_2) \leqslant x$ and $s_t(o_1) - Q \leqslant x < s_t(o_1)$ is not possible because $s_t(o_1) \leqslant s_t(o_2)$.

Case 9. $s_t(o_2) \leqslant x$ and $s_t(o_1) \leqslant x$. Then, $J_t(x + Q, o_2) - J_t(x, o_2) = H_t(x, o_2) - K_t \leqslant H_t(x, o_1) - K_t = J_t(x + Q, o_1) - J_t(x, o_1)$.

To conclude the induction argument, we show that for $t - 1$

$$\begin{aligned}H_{t-1}(x, o_2) &= K_{t-1} + G_{t-1}(x + Q) - G_{t-1}(x)\\ &\qquad + \alpha E[J_t(x_2 + Q, O_2) - J_t(x_2, O_2)]\\ &\leqslant K_{t-1} + G_{t-1}(x + Q) - G_{t-1}(x)\\ &\qquad + \alpha E[J_t(x_1 + Q, O_2) - J_t(x_1, O_2)]\end{aligned}$$

$$\begin{aligned}&\leqslant K_{t-1}+G_{t-1}(x+Q)-G_{t-1}(x)\\&\quad+\alpha E[J_t(x_1+Q,O_1)-J_t(x_1,O_1)]\\&=H_{t-1}(x,o_1).\end{aligned}$$

Note that $x_2 \leqslant x_1$ and $O_1 \leqslant O_2$ due to the state updates (Equations (1) and (2)) and $o_1 < o_2$. The inequalities are due to the second part of Lemma 3 and the induction argument, respectively.

For Part 2, we first show that $H_t(x, o+\epsilon e_j) \leqslant H_t(x, o+\epsilon e_{j+1})$. Note that from the third part of Lemma 4, this is true for $j=1$. Assume for an induction argument that it is true for j. Then, $s_t(x, o+\epsilon e_j) \geqslant s_t(x, o+\epsilon e_{j+1})$. Next, we show that this implies $J_t(x+Q, o+\epsilon e_j) - J_t(x, o+\epsilon e_j) \leqslant J_t(x+Q, o+\epsilon e_{j+1}) - J_t(x, o+\epsilon e_{j+1})$. To do so we consider three cases:

Case 1. $x < s_t(o+\epsilon e_{j+1}) - Q$. Then, $J_t(x+Q, o+\epsilon e_j) - J_t(x, o+\epsilon e_j) = H_t(x+Q, o+\epsilon e_j) - K_t \leqslant H_t(x+Q, o+\epsilon e_{j+1}) - K_t = J_t(x+Q, o+\epsilon e_{j+1}) - J_t(x, o+\epsilon e_{j+1})$.

Case 2. $s_t(o+\epsilon e_{j+1}) - Q \leqslant x < s_t(o+\epsilon e_j)$. Then, $J_t(x+Q, o+\epsilon e_j) - J_t(x, o+\epsilon e_j) \leqslant -K_t \leqslant J_t(x+Q, o+\epsilon e_{j+1}) - J_t(x, o+\epsilon e_{j+1})$.

Case 3. $s_t(o+\epsilon e_j) \leqslant x$. Then, $J_t(x+Q, o+\epsilon e_j) - J_t(x, o+\epsilon e_j) = H_t(x, o+\epsilon e_j) - K_t \leqslant H_t(x, o+\epsilon e_{j+1}) - K_t = J_t(x+Q, o+\epsilon e_{j+1}) - J_t(x, o+\epsilon e_{j+1})$.

From Equation (6) and the above three cases, $H_t(x, o+\epsilon e_{j+1}) \leqslant H_t(x, o+\epsilon e_{j+2})$ follows, concluding the induction. Part 3 for $j > 1$ follows immediately from Part 2 and Lemma 4 because $s_t(o+\epsilon e_j) - s_t(o) \leqslant s_t(o+\epsilon e_{j-1}) - s_t(o) \leqslant \cdots \leqslant s_t(o+\epsilon e_1) - s_t(o) \leqslant \epsilon$. □

Endnotes

1. On February 7, 2002, Richard Hunter from Dell announced this initiative during his presentation at the Stanford University Global Supply Chain Management Forum.
2. We use the term increasing and decreasing in the weak sense. Increasing means nondecreasing.
3. An inventory problem is said to be stationary if the cost and the demand distribution are time invariant.
4. $o_2 \geqslant o_1$ if and only if each element of o_2 is greater than or equal to the corresponding element of o_1.
5. For two positive functions f and g, $f = O(g)$ if there exists positive constants c and N such that $f(n) \leqslant cg(n)$ for all $n \geqslant N$.

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