Replenishment Strategies for Distribution Systems Under Advance Demand Information

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Customers with positive demand lead times place orders in advance of their needs. A portfolio of customers with different demand lead times gives rise to what we call *advance demand information*. We develop effective inventory policies for a distribution system to account for this information. In particular, we study a centralized system with one warehouse serving multiple retailers under advance demand information. The inventory manager replenishes the warehouse from an outside supplier. Units arriving to the warehouse are allocated to the retailers. To control this system, we develop a lower bound and proposed a close-to-optimal heuristic for which the optimality gap is on average 1.92%. We also provide a closed-form solution to approximate the system-wide inventory level. Using this explicit solution, the model and the heuristic, we investigate (1) the benefit of advance demand information, and its impact on allocation decisions, (2) the joint role of risk pooling and advance demand information, and (3) the system performance with respect to supplier and retailer lead times. We illustrate how advance demand information can be a substitute for lead times and inventory, and how it enhances the outcome of delayed differentiation.
(*Stochastic Inventory Systems; Advance Demand Information; Multi-Echelon; Distribution Systems; Supply Chain Management*)

1. Introduction and Literature Review

Advance demand information is obtained as customers place orders in advance of future demand requirements. Most inventory models treat customer demands as unanticipated events. Customers, however, often have different willingness to pay for the speed of delivery. A supply chain may improve its profit by satisfying customers who are willing to pay higher prices for shorter demand lead times, and by offering price discounts to those customers who are willing to accept longer demand lead times.[1] A portfolio of customers with different demand lead times results in what we call *advance demand information*. Under this scenario, the demand for any future period will be progressively revealed. This requires the inventory manager to follow an adaptive replenishment strategy where part of the orders is placed in anticipation of the customers' willingness to wait.

Obtaining and sensibly using advance demand information enables companies to be more responsive to customer needs and improve inventory management. Strategies to elicit advance demand information include price incentives and/or priority service to customers who book early (Chen 2000). Advance demand information is often shared by downstream supply-chain partners through contractual agreements (Harvard Business School 1991). An original equipment manufacturer, for example, places orders

[1] A customer who places his order l periods in advance is said to have a demand lead time l.

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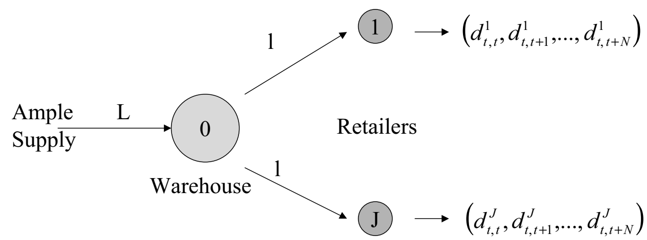
and updates these orders over a time window to its contract manufacturer (Stanford GSB 2001). An option to customize often induces willingness to wait and results in advance demand information. A classical example is Dell's distribution system (Dell 2000). Dell has a new initiative called "Intelligent Fulfillment," which allows for four different levels of response time to customer orders: (1) standard (conventional; 5-day promised order lead time), (2) value delivery (slower; lower shipping cost), (3) premium delivery (if the order is in by 8:00 a.m.; next-day delivery by 10:30 a.m.), and (4) precision delivery (specific date). This broader choice for delivery mode and timing will presumably enhance Dell's interaction with its customers, because customers have more choice in selecting delivery options and costs. This flexibility also allows Dell to provide better service levels, because they can better optimize their own inventory and manufacturing needs based on specific customer needs.[2] If appropriately used to derive inventory replenishment policies, advance demand information can be beneficial. In this paper, we show how to achieve this for a two-level distribution system.

Under the advance demand information model, the demand seen during any period t at retailer j is given by $d_t^j = (d_{t,t}^j, d_{t,t+1}^j, \ldots, d_{t,t+N}^j)$, where $d_{t,s}^j$ represents orders placed during period t for future period s at retailer j, and N is the information horizon beyond which the inventory manager does not collect advance demand information.

We analyze a periodic-review distribution system consisting of a central warehouse and J retailers as illustrated in Figure 1. Each retailer satisfies the demand of their portfolio of customers that collectively gives rise to the advance demand information vector d_t^j. The replenishment decisions are centralized and based on system-wide information; that is, there is a unique inventory manager who controls the entire distribution system.[3] The inventory manager replenishes retailers' inventory through a central warehouse. He orders from an outside supplier with ample stock to replenish the warehouse. Orders from the outside supplier and shipments to retailers arrive after fixed lead times L and l, respectively. We assume that the warehouse is a coordination and repackaging or cross-docking center so it does not hold inventory. Unsatisfied demand at each location is fully backordered. A penalty cost is charged on back orders at the retailers. Holding cost is charged on ending inventories at each retailer as well as on inventories in transit. These systems are also known as coupled systems (Zipkin 2000).

Figure 1 **Distribution System with Advance Demand Information**



Traditional research on distribution systems view the customer demand process as static and exogenous in that the demand is modeled as a stationary process. Federgruen (1992) provides a comprehensive review on this topic, including the papers by Eppen and Schrage (1981), Federgruen and Zipkin (1984a, b, c), Jackson (1988), and Schwarz (1989). As pointed out by these authors, there are two distinct advantages of having a central warehouse. First, it enables the inventory manager to negotiate lower procurement costs from the outside supplier due to aggregating volume. Second, it enables the manager to gain through *risk pooling* (or *statistical economies of scale*).[4] Risk pooling (for a system with a total replenishment lead time of $L+l$ periods) can be considered as the inventory manager's option to carry a single system inventory during the first L periods instead of J individual inventories. Notice that risk pooling mitigates the demand uncertainty during the first L periods, whereas advance demand information progressively eliminates the uncertainty in the system. Consider the case where *all* customers place orders N periods in

[2] On February 7, 2002, Richard Hunter from Dell announced this initiative during his presentation at Stanford University Global Supply-Chain Forum.

[3] Notice that under centralized control, information sharing and coordination issues are not our concern. Reviews of these topics are provided by Cachon (2001) and Chen (2001).

[4] A term coined by Eppen and Schrage (1981).

advance such that $N > L + l$. In this case, the retailers do not need to carry any inventory. A more intriguing case is a distribution system that serves a portfolio of customers with different demand lead times as in the Dell example.

In this paper, we take a step in the direction of incorporating a richer information structure obtained from customers. Our general goal is to understand how *advance demand information* affects the system—more specifically—how the system should be managed when advance demand information is available. We establish a close-to-optimal *state dependent* replenishment policy and show how it responds to the changes in customer demand. The state of the system is given by modified inventory positions for each retailer and observed demands beyond the retailer lead times.[5] Under this policy, the inventory manager monitors the system-wide *modified* inventory position. If it is less than a state dependent base-stock level, than he orders to bring the modified inventory position up to this level. Incoming materials to the warehouse are disaggregated and shipped to the retailers based on the solution of a *myopic* problem. We show that the base-stock level is an increasing[6] function of observed demand beyond the retailer lead times, and it is affected more by an increase in the observed demand that is closer to the current period. We examine the joint role of statistical economies of scale and advance demand information. We illustrate, for example, that advance demand information is a substitute for inventory as well as for supplier and retailer lead times. The results of this paper are necessary to understand the benefit of advance demand information and quantify its impact on the distribution systems to design effective incentives.

Establishing optimal policies for distribution systems, even in the absence of advance demand information, is computationally intractable. There are two approaches to solve this problem: approximation by *relaxation* as in Federgruen and Zipkin (1984a) and Aviv and Federgruen (2001a), and approximation by *restriction* as in Eppen and Schrage (1981) and Federgruen and Zipkin (1984b). The first approach considers relaxing a constraint set to obtain a simpler problem with lower-dimensional state space. It develops a heuristic based on this lower bound problem to solve the original problem. The second approach restricts the policy space to a class of policies and optimizes over this class under additional assumptions. The restriction approach, unlike the relaxation approach, does not guarantee any bound on the optimal solution. We employ both approaches here. Using the relaxation approach, we provide a close-to-optimal replenishment strategy. We also carry out a numerical study (170 instances) to gauge the sensitivity of the replenishment strategy with respect to problem parameters and to show that the optimality gap is fairly tight (on average 1.92%). Finally, using the restriction approach, we provide a closed-form solution for the system-wide base-stock level.

The numerical results and the closed-form solution enable us to address issues that arise in the design of production and distribution systems. It is important to understand, for example, how the benefit of advance demand information and risk pooling change as one locates the warehouse closer to the retailers or how reductions in lead times affect system performance. Some of these (re)designs may require investments in new processing plans or using faster but more expensive shipment modes. Marketing strategies such as the option to customize or promotions may result in advance demand information. It is important, therefore, to compare the benefits with the costs of these actions. Our model enables us to characterize the system performance (in terms of inventory management costs) with respect to such changes in the system. For example, in a distribution system with $L + l = 3$, moving the warehouse from a location closer to the supplier ($L = 0, l = 3$) to a location closer to the retailers ($L = 2, l = 1$) reduces costs 2.02% without advance demand information and 12.14% with advance demand information (see Table 6).

A group of researchers has addressed distribution systems under nonstationary demands. Erkip et al. (1990) and Güllü (1997) extend the model of Eppen

[5] Inventory positions are *modified* to subtract the known parts of the lead time demand. The observed demand information that is beyond the lead times is kept separately as a state variable.

[6] We use the term increasing in the weak sense; that is, increasing means nondecreasing.

and Schrage (1981), respectively, incorporating a correlation structure through the use of an index variable that follows autoregressive time series process and a forecast process. All these authors restrict the policy space to a *constant*, base-stock level, hence these policies do not fully use the information obtained through the demand process. In contrast, Song and Zipkin (1993) incorporate Markov modulated demand process in multiechelon inventory systems and evaluate the performance of a *state-dependent* base-stock policy that depends on the underlying Markov process. In a recent paper, Aviv and Federgruen (2001b) incorporate a Bayesian framework to the demand process. By doing so, they introduce the additional benefit, *learning effect*, of having a central warehouse. The ability to obtain information about the demand during the first L periods enables updating the demand process, hence resulting in improved allocation to retailers.[7]

The distribution system described here can also be interpreted as a multi-item production/distribution system with a common intermediate product. In this interpretation, the warehouse represents the differentiation point. During the first phase of the production period, L, a common batch is produced. At the end of this period, the manager must decide on how much of each differentiated item to produce from the batch of the common intermediate product. This interpretation forms the basis of postponement strategies; see the papers by Lee et al. (1993), Aviv and Federgruen (2001a, b), and the literature therein. A special case of this problem, where $L = 0$ was studied by Veinott (1965). He investigates conditions that ensure the optimality of base-stock policies. Through a numerical example, we illustrate how advance demand information increases the benefit gained through postponement strategies.

The relaxation approach yields a lower bound problem that can be interpreted as a single-stage inventory control problem. A number of researchers incorporate the dynamic nature of demand, to single location inventory problems (Johnson and Thompson 1975, Song and Zipkin 1993, Heath and Jackson 1994, Gallego and Özer 2001, Aviv 2002). For a more detailed review on the use of demand information, we refer the reader to Gallego and Özer (2002).

[7] The authors discuss this in the context of a multi-item system.

The rest of this paper is organized as follows. In §2, we introduce the demand model and the dynamic programming formulation. In §3, we apply the relaxation approach to obtain a lower bound model. We establish the optimality of a state-dependent base-stock policy first for the $L = 0$ case followed by $L > 0$. We do this to separate the effect of *advance demand information* from that of *risk pooling*. In §4, we propose a heuristic to control ordering and shipping decisions for the system. In §5, we conduct a numerical study of several problem instances. First, we report on the performance of the heuristic. Next, we provide insights into a set of design problems and the benefits of advance demand information. In §6, we apply the restriction approach and state our assumptions to provide a closed-form solution for the system-wide base-stock level. In §7, we conclude and suggest directions for future research. All the proofs are in the appendix.

2. Model Formulation

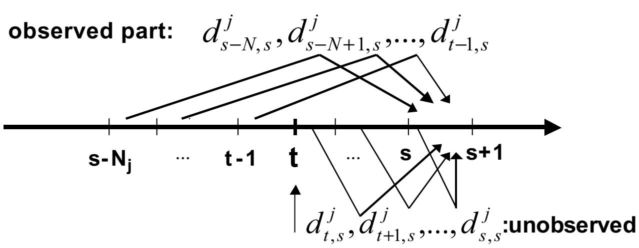
During any period t, retailer j's customers either purchase the product or place an order for a future period $s \in \{t, \ldots, t+N\}$. From the view point of the inventory manager, the demand observed at each retailer j is a vector: $d_t^j \equiv (d_{t,t}^j, \ldots, d_{t,t+N}^j)$, where $d_{t,s}^j$ is the demand for period s observed during period t and $N < \infty$ is the length of the *information horizon*.

We assume that these demand vectors are completely revealed at the end of period t. Our analysis requires that the demand vector across time be independent, but could be nonstationary. The demand vector across retailers and within the demand vector could also be correlated. We do not require differentiability and, hence the results apply to both the discrete and the continuous demand case.

Under this demand model at the beginning of each period t, the demand to be realized at retailer j during a future period $s \in \{t, \ldots, t+N-1\}$ can be divided into two parts as illustrated in Figure 2: the *observed* and known part $o_{t,s}^j \equiv \sum_{r=s-N}^{t-1} d_{r,s}^j$ and the *unobserved* and unknown part $u_{t,s}^j \equiv \sum_{r=t}^{s} d_{r,s}^j$.

Notice that our demand model can incorporate random order quantities and desired fulfillment dates.

Figure 2 Observed and Unobserved Part of the Demand



Let N_t^j be the number of customers that arrive to retailer j at period t. Assume that customer k demands a random quantity $Q_{t,k}$ to be delivered at a random fulfillment date $s_{t,k} \in \{t, \ldots, t+N\}$. Then, $d^j_{t,s} = \sum_{k=1}^{N_t} Q_{t,k} I(s_{t,k} = s)$, where $I(\cdot)$ is an indicator function. Hariharan and Zipkin (1995) refer to $s_{t,k} - s$ as the demand lead time and study a single location problem, where *all* customers have *constant* demand lead times. Our demand model can be viewed as a periodic review generalization of Hariharan and Zipkin (1995) to the case of random demand lead times.

The sequence of events is as follows: (1) the inventory manager reviews I_t^j on-hand inventory, B_t^j back orders, the pipeline inventory, *and* $o^j_{t,s}$, the observed part of the demand for periods $s \in \{t, t+1, \ldots, t+N-1\}$ for each retailer.[8] He decides on (i) whether or not to place an order, $w_t \geq 0$, from an outside supplier, and (ii) how to allocate the incoming batch among the retailers, $z_j \geq 0$. It takes $L \geq 0$ periods to receive the orders from an outside supplier and $l > 0$ periods to receive the shipments allocated to retailers. The total shipment to the retailers is equal to the incoming orders to the warehouse, $\sum_{j=1}^J z_t^j = w_{t-L}$ because the warehouse does not hold any inventory. The manager incurs a linear ordering and shipping cost $c_t^0 w_t + \sum_{j=1}^J c_t^j z_t^j$. (2) The shipments placed at period $t-l$ arrive at the retailers. (3) Demand vectors for period t at each retailer are realized. The demand for period t is satisfied from on-hand inventory, otherwise it is backlogged. (4) The manager incurs holding and penalty cost at each retailer based on the end-of-period net inventory.

[8] $o^j_{t,s} \equiv 0$ for $s \geq t+N$ because we do not observe demand information beyond the information horizon.

Notice that the allocation decision has an impact on retailer j's holding and penalty cost only after l periods. The inventory manager should, therefore, protect the retailer against the lead time demand, which is defined as the total demand realized over periods $\{t, t+1, \ldots, t+l\}$. We use the standard accounting mechanism and charge period t with the expected inventory costs incurred at period $t+l$. The *net* inventory at the end of period $t+l$ is given by inventory on-hand at period t *plus* the inventory in the pipeline, *including* the order just allocated to the retailer *minus* the backlogs *minus* the realized demand during the lead time. Notice that unlike the classical case (where the information horizon $N=0$), we can divide the lead time demand at retailers into two parts: the observed part and the unobserved part, $\sum_{s=t}^{t+l} o^j_{t,s} + \sum_{s=t}^{t+l} u^j_{t,s}$. We designate a set of new inventory variables to account for the known part. That is, we define

$$
\begin{aligned}
x_t^j &= \textit{modified} \text{ inventory position } \textit{before} \\
&\quad \text{shipment decision} \\
&= \left(I_t^j + \sum_{s=t-l}^{t-1} z_{js} - B_t^j \right) - \sum_{s=t}^{t+l} o^j_{t,s} \quad \text{for all } j \in \{1, \ldots, J\}; \\
y_t^j &= \textit{modified} \text{ inventory position } \textit{after} \\
&\quad \text{shipment decision} \\
&= x_t^j + z_t^j \quad \text{for all } j \in \{1, \ldots, J\}.
\end{aligned}
$$

These variables differ from classical inventory variables in that they subtract the observed part of the lead time demand, hence the name *modified*. This enables us to reduce the dimension of the state space of the dynamic programming algorithm.

The expected holding and penalty cost charged to period t for retailer j is given by

$$
\tilde{G}_t^j(y_t^j) = \alpha^l E g_t^j \left(y_t^j - \sum_{s=t}^{t+l} u^j_{t,s} \right),
$$

where α is the discount factor and the expectation is with respect to the unobserved part of the demand realized during periods t through $t+l$. For all t and j, we assume that g_t^j is convex, that $\tilde{G}_t$ exists, and that $\lim_{|x|\to\infty} \tilde{G}(x) = \infty$ (these are standard assumptions in the inventory literature). All these assumptions, for example, are satisfied for a linear holding cost h_t^j per item, and a linear penalty cost p_t^j per back order. The expected holding and penalty cost charged to period t for retailer j for this example would be $\tilde{G}_t^j(y_t^j) = \alpha^l E\{h_t^j[y_t^j - \sum_{s=t}^{t+l} u_{t,s}^j]^+ + p_t^j[y_t^j - \sum_{s=t}^{t+l} u_{t,s}^j]^-\}$.

The state of the system is given by three vectors:

$$x_t\colon (x_t^1, \ldots, x_t^J), \quad v_t\colon (w_{t-L}, \ldots, w_{t-1}),$$
$$o_t\colon (o_t^1, o_t^2, \ldots, o_t^J),$$

where o_t^j is the observed demand information *beyond* retailer j's lead time; that is, $o_t^j = (o_{t,t+l+1}^j, \ldots, o_{t,t+N-1}^j)$. Notice that the vector o_t^j exists only for $N > l+1$. This vector holds the observed part of the demand information that is not subsumed in the modified inventory position. The vector v_t is the pipeline inventory between the supplier and the warehouse.

After making the order and shipment decisions and observing the demand vectors $d_t^j = (d_{t,t}^j, \ldots, d_{t,t+N}^j)$ at each retailer, the manager updates the modified inventory positions by

$$x_{t+1}^j = x_t^j + z_t^j - d_{t,t}^j - \sum_{s=t+1}^{t+l+1} d_{t,s}^j - o_{t,t+l+1}^j \tag{1}$$

and the vector of observed demand beyond the lead time by

$$o_{t+1}^j = (o_{t+1,t+l+2}^j, \ldots, o_{t+1,t+N}^j), \tag{2}$$

where $o_{t+1,s}^j = o_{t,s}^j + d_{t,s}^j$.

The solution to the following dynamic programming algorithm minimizes the cost of managing this two-level distribution system for a finite horizon with $T-t$ periods remaining to the termination:

$$J_t(x_t, v_t, o_t) = \min_{\{w_t, y_t\colon\ \sum_{j=1}^J (y_t^j - x_t^j) = w_{t-L},\, y_t \ge x_t,\, w_t \ge 0\}} \Big\{ c_t^0 w_t + \sum_{j=1}^J G_t^j(y_t^j) + \alpha E J_{t+1}(x_{t+1}, v_{t+1}, o_{t+1}) \Big\}, \tag{3}$$

where $G_t^j(y_t^j) = (c_t^j - \alpha c_{t+1}^j) y_t^j + \tilde{G}_t^j(y_t^j)$ and $J_{T+1}(\cdot,\cdot,\cdot) \equiv 0$ and the expectation is with respect to the demand vectors d_t^j. The formal construction of this dynamic program is similar to Gallego and Özer (2001).

We remark that if the inventory manager wishes to incorporate advance demand information only up to the next $l+1$ periods for all retailers, then the state space would be similar to that of classical distribution systems (as in Federgruen and Zipkin 1984a). All proposed heuristics to solve the classical distribution system and the reported performance measures would apply to this special case. The inventory manager would only need to update his inventory position to account for the observed part and use *modified* inventory positions instead. From this point, on we consider only the problem where $N > l+1$.

3. Lower Bound Model

The state space for the dynamic program in Equation (3) is of dimension $J + L + J(N-l-1)^+$. It is impractical to deal with such a large state space. Even in the absence of advance demand information, the classical distribution system runs into the problem of "dimensionality." In this section, we develop a lower bound approximation that has a lower-dimensional state space by relaxing the constraint $y_t \ge x_t$. We conclude by establishing the optimality of a *state dependent* base-stock policy for the lower bound problem.

Relaxing the constraint $y_t \ge x_t$ means that shipments can be a negative quantity as long as the *total* shipments from the warehouse is equal to the incoming orders. This can be interpreted as: If the inventory (pipeline and on-hand) at some retailers are high, some of this excess can be transferred to other retailers at no cost. Similarly, if the observed part of the lead time demand (this could be interpreted as retailers' commitment) is excessive, part of this demand can be satisfied through other retailers that have high-modified inventory position at no cost. Under this relaxation, all retailers collapse into an aggregate retailer. The state space of this new relaxed problem will be based on aggregate quantities. Let

$$D_{t,s} = \sum_{j=1}^J d_{t,s}^j, \quad X_t = \sum_{j=1}^J x_t^j,$$
$$O_{t,s} = \sum_{j=1}^J o_{t,s}^j, \quad U_{t,s} = \sum_{j=1}^J u_{t,s}^j.$$

We use the convention of capital letters to indicate the aggregation across all retailers. The state space of the relaxed dynamic program is given only by the aggregate quantities X_t, v_t, and $O_t = (O_{t,\,t+l+1}, \ldots, O_{t,\,t+N-1})$. The dynamic programming recursion below solves this aggregate retailer problem and provides a lower bound for the original distribution system.

Theorem 1. *For all (x_t, v_t, o_t), the relaxation results in a dynamic programming algorithm with the state space aggregated across all retailers and is given by*

$$\begin{aligned} V_t(X_t, v_t, O_t) &= \min_{\{w_t,\, Y_t:\ w_t \geq 0,\, Y_t = X_t + w_{t-L}\}} \big\{ c_t^0 w_t + R_t(Y_t) \\ &\qquad + \alpha E V_{t+1}(X_{t+1}, v_{t+1}, O_{t+1}) \big\}, \end{aligned} \tag{4}$$

where $V_{T+1} \equiv 0$, the update for X_t is given by $X_{t+1} = Y_t - \sum_{s=t}^{t+l+1} D_{t,s} - O_{t,\,t+l+1}$ and

$$R_t(Y_t) = \left\{ \min_{\forall y_t^j} \sum_{j=1}^{J} G_t^j(y_t^j) : \sum_{j=1}^{J} y_t^j = Y_t \right\}. \tag{5}$$

The solution to this DP is a lower bound to the solution of $J_t(x_t, v_t, o_t)$ defined in (3). *In addition, $R_t(Y_t)$ is convex and $\lim_{|Y_t| \to \infty} R_t(Y_t) = \infty$.*

Y_t represents the sum of all retailers' modified inventory positions, plus the order that has just arrived to the warehouse at period t. Hence, function R_t is the minimum cost of managing retail inventories given Y_t, assuming that the retail inventories can be rebalanced instantaneously and at no cost. We differentiate next the two cases: a negligible supplier lead time $L = 0$ and a positive supply lead time $L > 0$. This helps us determine the impact of advance demand information on the system independent of the effect of risk pooling.

3.1. Lower Bound Problem When $L = 0$

The distribution system without advance demand information and with zero supply lead time benefits only through its ability to negotiate lower procurement prices (Eppen and Schrage 1981). The dynamic program for this case is given by

$$V_t(X_t, O_t) = -c_t^0 X_t + \min_{Y_t \geq X_t} H_t(Y_t, O_t),$$

where $H_t(Y_t, O_t) \equiv c_t^0 y + R_t(y) + \alpha E V_{t+1}(x_{t+1}, O_{t+1})$, $V_{T+1} \equiv 0$, and the update is $X_{t+1} = Y_t - \sum_{s=t}^{t+l+1} D_{t,s} - O_{t,\,t+l+1}$.

In Theorem 2, we establish first, the optimality of *state-dependent* base-stock policy and characterize its behavior. Under this policy, the warehouse orders whenever the system-wide modified inventory position falls below a base-stock level. We show that the optimal base-stock level is increasing in observed demand beyond the retailer lead time l. In particular, an increase in the observed demand beyond the retailer lead time that is closer to the current period increases the base-stock level more than the one that is far out in the future. Furthermore, this increase in the base-stock level is never more than the increase in the observed demand. To show this, we define e_j as the $(N-l-1)$-dimensional unit vector whose jth element is 1. By adding ϵe_j to the vector, O_t is equivalent to increasing the cumulative observed demand in the period that is j periods beyond the lead time by ϵ units, that is $O_{t,\,t+l+j} + \epsilon$.

Theorem 2. *The following statements are true for any vector O_t, $\epsilon > 0$, and for all t:*

(1) *$H_t(x, O_t)$ is convex,*

(2) *A state dependent base-stock policy is optimal. The base-stock level is given by*

$$y_t^*(O_t) \equiv \min\{y : H_t(x, O_t) \geq H_t(y, O_t) \text{ for all } x \neq y\},$$

(3) *$V_t(x, O_t) + c_t^0 x$ is an increasing convex function,*

(4) *$y_t^*(O_t + \epsilon e_j) \geq y_t^*(O_t + \epsilon e_{j+1})$ for $j = 1, \ldots, N-l-2$,*

(5) *$y_t^*(O_t + \epsilon e_j) - y_t^*(O_t) \leq \epsilon$ for $j = 1, \ldots, N-l-1$, and*

(6) *$y_t^*(O_t) \geq y_t^*(O_t')$ for all $O_t' \geq O_t$.*[9]

These results delineate the distribution systems response with respect to observed demands beyond the retailer lead time. The system responds by increasing the base-stock level if there is demand to come in the future. The manager is more likely to order in that case and when he does, he orders a larger quantity. These kinds of monotonicity results also limit the search for optimal policy parameters, hence improving the computational work required to solve large-scale problems.

[9] For two vectors O_t' and O_t, $O_t' \geq O_t$ if and only if each element of O_t' is greater than or equal to the corresponding element of O_t.

3.2. Lower Bound Problem when $L > 0$

We apply an accounting scheme to subtract pipeline inventory from the supplier to the warehouse. This scheme was introduced by Veinott (1965, 1980) in the context of single location problems and was applied to distribution systems first by Federgruen and Zipkin (1984a). If we initiate an order from the outside supplier at the beginning of period t, it will arrive to the warehouse by the beginning of period $t+L$. Consequently, the total modified inventory position, plus the order that has just arrived to be rebalanced at period $t+L$ is given by

$$Y_{t+L} = \sum_j x^j_{t+L} + w_t$$
$$= \underbrace{X^\Delta_t + w_t}_{Y^\Delta_t} - \sum_{r=t}^{t+L-1} \sum_{s=r}^{r+l+1} D_{r,s} - \sum_{s=t}^{t+L-1} O_{s,s+l+1}.$$

We obtain the second equality by substituting x^j_{t+L} for the update (1) and w_{s-L} for $\sum_j z^j_s$, *and* defining the *system-wide modified* inventory position at period t as $X^\Delta_t = X_t + \sum_{s=t}^{t+L-1} w_{s-t}$. The dynamic program is given by

$$\tilde{V}_t(X^\Delta_t, O_t) = -c^0_t X^\Delta_t + \min_{Y^\Delta_t \geq X^\Delta_t} H_t(Y^\Delta_t, O_t), \qquad (6)$$

where $H_t(y, O_t) = c^0_t y + \alpha^L E R_{t+L}(y) + \alpha E \tilde{V}_{t+1}(X^\Delta_{t+1}, O_{t+1})$, $X^\Delta_{t+1} = y - \sum_{s=t}^{t+l+1} D_{t,s} - O_{t,t+l+1}$, and $\tilde{V}_{T+1} \equiv 0$. We have

LEMMA 1. $\tilde{V}_t(X^\Delta_t, O_t) = V_t(X_t, v_t, O_t) + r_t(X_t, v_t, O_t)$.

This result states that the algorithm in (6) is similar to the one in (4), except the constant term r_t, which is defined in the appendix. This term is independent of order and shipment decisions, hence it can be dropped for optimization purposes. We remark that Theorem 2 also applies to the above dynamic programming algorithm.

4. Proposed Heuristic

We decide on the aggregate order quantity based on a state-dependent base-stock level obtained from the solution of the lower bound problem. The aggregate order quantity is given by $w_t(O_t) \equiv y^*_t(O_t) - X^\Delta_t$. This quantity depends on the observation beyond the aggregate retailer lead time as elaborated in Theorem 2. In this sense, the lower bound problem makes full use of advance demand information. Next, we decide on how to allocate the incoming order, w_{t-L}. We allocate this order to retailers based on the solution of the following allocation problem:

$$\min \left\{ \sum_{j=1}^{J} G^i_t(y^i_t) : \sum_{j=1}^{J} (y^j_t - x^j_t) = w_{t-L}, \, y_t \geq x_t \right\}. \qquad (7)$$

This allocation is referred to as *myopic* for two reasons. First, it minimizes the total expected cost of managing the retailer inventories at the end of period $t+l$ (the period when the allocations are available for the retailer) and ignores the impact of this allocation on other periods. Second, it does not take into consideration the observed demand information beyond the retailers' lead time (unlike the lower bound). Notice that this is a feasible solution to the original problem, hence it is an upper bound.

In our numerical study, we use a greedy algorithm both to solve the myopic allocation and to evaluate $R_t(Y_t)$ defined in Equation (5). To solve the allocation problem, we start with $y^i_t = x^i_t$ and allocate one unit at a time to the jth retailer with the smallest current value of the first difference (that is, $\min_j \{G^j_t(y+1) - G^j_t(y)\}$) until all w_{t-L} is allocated. To evaluate $R_t(Y_t)$, we start with $R_t(\sum_{j=1}^{J} {}^*y^j_t)$, where ${}^*y^j_t$ is the minimum of $G^j_t(\cdot)$ and allocate the difference $Y_t - Y^*_t$, if positive, one unit at a time to the jth retailer with the smallest current value of the first difference. Otherwise, we reduce the amount allocated one unit at a time from the jth retailer with the largest current value of the first difference (that is, $\max_j \{G^j_t(y) - G^j_t(y-1)\}$). Similar algorithms are used by Fox (1966), Federgruen and Zipkin (1984a, b), Zipkin (2000), and Aviv and Federgruen (2001a). We remark that a capacitated distribution system, in which a warehouse is allowed to process up-to a limited quantity, can be solved by using the replenishment policy in Özer and Wei (2001) together with a myopic allocation.

5. Numerical Study

First, we compare the solution of the lower bound (LB) problem to the solution of the proposed heuristic (upper bound (UB)). We report the difference as percentage error, $\epsilon\% = (UB - LB)/LB$, which is a measure of the suboptimality for the heuristic. A small value indicates that the heuristic is close to optimal and that the lower bound is accurate. We use a backward induction algorithm to solve the LB problem (6) and to obtain the optimal system base-stock level $y_t^*(O_t)$ and the associated cost. All cases considered have linear holding, back ordering costs, and uniform rates. We simulate the system to estimate the cost of the proposed heuristic. We run several replications to have a 95% confidence interval. We compute both the lower bound and the heuristic for a finite horizon of length T. Classical literature on distribution systems minimize average cost as an objective, but notice that this is an approximation for a finite horizon system. We set each retailers' initial inventory to $I_0^j = \lfloor (y_0^*(O_0))/J \rfloor$ and $o_0^j \equiv 0$. We compare the simulation outcome with the cost of the LB for which the initial state is $X_0^\Delta = \sum_j I_0^j$. We conclude by providing some insights into strategic questions that may arise in the design of distribution systems.

We use Poisson random variables to model the demand vector. That is, $d_{t,s}^j$ is Poisson with mean λ_{s-t}^j. In many practical situations, demand behaves like a Poisson process, in particular, when the demand comes from many small nearly independent sources such as individual customers spread across a large region (Zipkin 2000, p. 179). We discuss the possible effects of other distributions on our numerical results. The following example clarifies how we model different advance demand information scenarios.

Example. Assume that information horizon for a two-retailer distribution system is two periods ($N = 2$). Let t be the current period. The demand vector to be realized during period t for retailer j is $(d_{t,t}^j, d_{t,t+1}^j, d_{t,t+2}^j)$. The total demand that is going to be realized during period $s > t+2$ is equal to $d_{s-2,s}^j + d_{s-1,s}^j + d_{s,s}^j$, which is Poisson with mean $\lambda_2^j + \lambda_1^j + \lambda_0^j$. By increasing this sum, one increases the variance of the demand seen by retailer j at each period. During period $s-2$, $d_{s-2,s}^j$ is realized and the remaining uncertainty is only due to $d_{s-1,s}^j + d_{s,s}^j$. Now consider the following scenarios: Under the first scenario, assume that the retailers are incapable of obtaining demand information in advance. To reflect this case, we model the demand vector by Poisson random variables with means $\lambda_0^j > 0$ and $\lambda_1^j = \lambda_2^j \equiv 0$. Now assume that all customers place their demand one period in advance, that is $\lambda_0^j = \lambda_2^j \equiv 0$ and $\lambda_1 > 0$. Finally, assume that all customers place orders two periods in advance. This case is modeled by $\lambda_0^j = \lambda_1^j \equiv 0$ and $\lambda_2^j > 0$. Any other advance demand information scenario can be similarly modeled.

We consider the number of retailers, advance demand information scenario, supplier lead time L, retailer lead time l, length of the planning horizon T, and the cost structure h, p, c to be the drivers of our numerical study. Throughout the numerical study, we set $N = l+2$. For this case, the LB problem is two-dimensional and O_t is a scalar that is equal to $D_{t-1,t+l+1}$. We use the following set of parameters:

$h = 0.05, 1, 2$	$p = 1, 5, 19$	$c = 0.5, 10, 30$	$T = 50, 75$
$L = 0, 1, 2$	$l = 1, 2, 3$	$J = 2, 5, 6, 8, 12$	$N = l+2$

We study a total of 170 instances. The first part of our numerical study addresses 110 instances in which the retailers are identical. The second part addresses 16 instances in which retailers differ in their advance demand information scenarios. We refer to this as *information imbalance*. The final part (44 instances) focuses on distribution system design experiments.

5.1. Performance of the Heuristic

The average optimality gap is 1.92% for the first group (110 instances). Within this group, the first set consists of 44 experiments in which the retailer lead time is $l = 1$ (see Table 1). The second set consists of 34 experiments in which $l = 2$ (see Table 2). Within each set, we address a subset of distribution systems with 2, 5, and 8 retailers. For each of these subsets, we address different advance demand information scenarios starting with the case of no advance demand information ($\lambda_0 > 0, 0, \ldots, 0$). To observe the impact of the planning horizon on the error term, we divide this group into 90 experiments with $T = 50$ periods and 20 experiments with $T = 75$ periods (see Table 3).

The error term is relatively insensitive with respect to the number of retailers (see Tables 1 and 2). The average error term for a system with two retailers is 2.42%, whereas for five retailers, it is 2.46%. On the

Table 1 **Identical Retailers with $l = 1$ ($T = 50$, $h = 1$, $p = 19$, $c = 10$)**

		$L=0, l=1$				$L=1, l=1$			
# ret	$(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$	$y_t(0)$	LB	UB	%	$y_t(0)$	LB	UB	%
2	(1, 0, 0, 0)	10	2,334.37	2,339.35 ± 8.36	0.21	12	2,366.46	2,384.94 ± 7.01	0.78
	(0, 1, 0, 0)	6	2,246.96	2,262.36 ± 8.60	0.69	8	2,301.18	2,312.45 ± 7.35	0.49
	(0, 0, 1, 0)	0	2,019.98	2,080.94 ± 8.15	**3.02**	5	2,176.00	2,268.14 ± 7.96	4.23
	(0, 0, 0, 1)	0	1,979.98	2,021.17 ± 8.03	2.08	0	1,999.98	2,021.73 ± 8.13	1.09
2	(2, 0, 0, 0)	16	4,464.43	4,484.98 ± 16.50	0.46	20	4,510.93	4,583.41 ± 13.39	1.61
	(0, 2, 0, 0)	10	4,350.36	4,386.63 ± 16.28	0.83	14	4,411.44	4,496.25 ± 14.46	1.92
	(0, 0, 2, 0)	0	4,040.00	4,154.84 ± 16.39	2.84	8	4,250.93	4,407.20 ± 15.85	3.68
	(0, 0, 0, 2)	0	3,960.00	4,036.72 ± 16.04	1.94	0	4,000.00	4,036.74 ± 16.04	0.92
2	(3, 0, 0, 0)	20	6,578.31	6,633.18 ± 27.76	0.83	27	6,633.07	6,770.31 ± 19.22	2.07
	(0, 3, 0, 0)	12	6,430.64	6,506.48 ± 25.21	1.18	19	6,508.76	6,550.34 ± 21.95	0.64
	(0, 0, 3, 0)	0	6,060.00	6,237.25 ± 23.68	2.92	10	6,317.61	6,532.37 ± 22.99	3.40
	(0, 0, 0, 3)	0	5,940.00	6,056.99 ± 23.39	1.97	0	6,000.00	6,056.99 ± 23.99	0.95
5	(1, 0, 0, 0)	25	5,831.99	5,844.02 ± 20.78	0.21	28	5,880.04	5,969.95 ± 18.81	1.53
	(0, 1, 0, 0)	15	5,613.31	5,651.03 ± 21.49	0.67	20	5,676.92	5,748.55 ± 18.69	1.26
	(0, 0, 1, 0)	0	5,050.00	5,199.71 ± 20.46	2.96	9	5,282.65	5,560.32 ± 19.87	5.26
	(0, 0, 0, 1)	0	4,950.00	5,050.39 ± 20.05	2.03	0	5,000.00	5,050.39 ± 20.04	1.01
5	(2, 0, 0, 0)	40	11,157.24	11,208.12 ± 41.14	0.46	48	11,220.13	11,445.59 ± 34.36	2.01
	(0, 2, 0, 0)	25	10,872.45	10,958.28 ± 40.66	0.79	34	10,938.67	11,275.13 ± 36.47	3.08
	(0, 0, 2, 0)	0	10,100.00	10,382.87 ± 40.93	2.80	15	10,428.50	10,819.59 ± 39.90	3.75
	(0, 0, 0, 2)	0	9,900.00	10,085.02 ± 40.25	1.87	0	10,000.00	10,085.02 ± 40.25	0.85
8	(1, 0, 0, 0)	40	9,329.16	9,346.49 ± 33.23	0.19	45	9,385.09	9,530.97 ± 26.54	1.55
	(0, 1, 0, 0)	24	8,979.10	9,039.91 ± 34.35	0.68	32	9,055.15	9,194.52 ± 29.95	1.54

Table 2 **Identical Retailers with $l = 2$ ($T = 50$, $h = 1$, $p = 19$, $c = 10$)**

		$L=0, l=2$				$L=1, l=2$			
# ret	$(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$	$y_t^*(0)$	LB	UB	%	$y_t^*(0)$	LB	UB	%
2	(2, 0, 0, 0, 0)	20	4,562.49	4,631.83 ± 13.5	1.52	25	4,596.42	4,661.59 ± 13.04	1.42
	(0, 2, 0, 0, 0)	16	4,464.43	4,619.68 ± 14.42	3.48	20	4,510.59	4,589.65 ± 13.46	1.75
	(0, 0, 2, 0, 0)	10	4,350.36	4,465.10 ± 16.50	2.64	14	4,411.44	4,498.63 ± 14.49	1.98
	(0, 0, 0, 2, 0)	0	4,040.00	4,346.72 ± 17.92	7.59	8	4,250.93	4,405.26 ± 15.82	3.63
	(0, 0, 0, 0, 2)	0	3,960.00	4,221.89 ± 17.54	6.61	0	4,000.00	4,034.27 ± 16.87	0.86
2	(3, 0, 0, 0, 0)	28	6,683.13	6,856.66 ± 20.07	2.60	35	6,727.57	6,861.76 ± 18.70	1.99
	(0, 3, 0, 0, 0)	20	6,578.31	6,776.52 ± 22.06	3.01	27	6,633.07	6,774.65 ± 20.25	2.13
	(0, 0, 3, 0, 0)	12	6,430.64	6,695.28 ± 24.43	4.12	19	6,508.76	6,680.26 ± 21.95	2.63
	(0, 0, 0, 3, 0)	0	6,060.00	6,518.77 ± 27.12	7.57	10	6,317.61	6,526.16 ± 24.05	3.30
	(0, 0, 0, 0, 3)	0	5,940.00	6,332.72 ± 26.72	6.61	0	6,000.00	6,051.19 ± 26.13	0.85
5	(2, 0, 0, 0, 0)	50	11,401.62	11,576.24 ± 34.06	1.53	62	11,429.87	11,625.79 ± 33.08	1.71
	(0, 2, 0, 0, 0)	40	11,157.24	11,546.42 ± 36.45	3.49	48	11,220.13	11,462.04 ± 34.91	2.16
	(0, 0, 2, 0, 0)	25	10,872.45	11,159.42 ± 41.72	2.64	34	10,938.67	11,281.37 ± 36.93	3.13
	(0, 0, 0, 2, 0)	0	10,100.00	10,861.80 ± 44.61	6.60	15	10,428.50	10,820.00 ± 40.30	3.75
	(0, 0, 0, 0, 2)	0	9,900.00	10,553.79 ± 44.61	6.60	15	10,000.00	10,086.72 ± 40.78	0.87
8	(1, 0, 0, 0, 0)	48	9,589.64	9,621.13 ± 26.95	0.33	57	9,633.83	9,678.64 ± 28.23	0.47
	(0, 1, 0, 0, 0)	40	9,329.16	9,570.88 ± 27.98	2.59	45	9,385.09	9,560.79 ± 28.84	1.87

Table 3 **Identical Retailers Where** $T = 75$ ($h = 1, p = 19, c = 10$)

		$L=0, l=1$				$L=1, l=1$			
# ret	$(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$	$y_t(0)$	LB	UB	%	$y_t(0)$	LB	UB	%
2	(1, 0, 0, 0)	10	3,506.85	3,511.59 ± 12.05	0.14	12	3,552.66	3,590.62 ± 10.44	1.07
	(0, 1, 0, 0)	6	3,370.28	3,385.43 ± 12.90	0.45	8	3,449.15	3,455.32 ± 11.21	0.18
2	(2, 0, 0, 0)	16	6,698.05	6,713.59 ± 24.56	0.23	20	6,764.51	6,882.15 ± 19.93	1.74
	(0, 2, 0, 0)	10	6,522.85	6,552.81 ± 24.12	0.46	14	6,611.16	6,742.11 ± 21.52	1.98
	(0, 0, 2, 0)	0	6,040.00	6,150.45 ± 24.14	1.83	8	6,367.74	6,601.45 ± 23.58	3.67
	(0, 0, 0, 2)	0	5,960.00	6,030.73 ± 23.75	1.19	0	6,000.00	6,030.73 ± 23.75	0.51
		$L=0, l=2$				$L=1, l=2$			
2	(1, 0, 0, 0)	12	3,600.28	3,611.36 ± 10.08	0.31	15	3,642.62	3,690.84 ± 9.94	1.32
	(0, 1, 0, 0)	10	3,506.85	3,593.78 ± 10.43	2.48	12	3,552.67	3,591.41 ± 10.07	1.09
2	(2, 0, 0, 0)	20	6,839.83	6,930.89 ± 20.24	1.33	25	6,893.59	6,992.11 ± 19.58	1.43
	(0, 2, 0, 0)	16	6,698.05	6,914.66 ± 21.67	3.23	20	6,764.51	6,888.76 ± 20.22	1.84

other hand, increasing the retailer lead times seems to increase the error term. The average gap is 1.67% for the set $l = 1$ and it is 3.09% for the set $l = 2$. We believe that this is due to the larger uncertainty introduced to the system by longer retailer lead time. Recall that the heuristic cannot reallocate the imbalances in modified inventory positions at each retailer, whereas the lower bound can. The comparison of the experiments in Table 3 with their counterparts in Tables 1 and 2 shows that the error term is decreasing as we increase the planning horizon. This suggests that under average cost criterion, the heuristic would more likely better perform.

We observe an increase followed by a decrease in the error term as the inventory manager obtains demand information further in advance. Consider, for example, the initial set of experiments in Table 1. The error 3.02% corresponds to the case where retailers learn their demand two periods in advance. The LB problem makes better use of advance demand information than the UB (the heuristic), hence we observe an increase in the error term (as discussed in more detail after Equation (7)). In Table 4, in which we address systems with retailers facing different advance demand information scenarios, the average error is 3.71%. The maximum error was 13.2% for a distribution system $L = 0, l = 2$ with two retailers, one of which has no advance demand information while the other obtains demand information four periods in advance. This case represents the maximum possible information imbalance that this distribution system can face. We also observe that the error term decreases as the number of balanced retailers in the system increases. One final observation is that as we increase the mean which is also the variance, the optimality gap tends to increase. Together with the numerical results reported in Aviv and Federgruen (2001a) and Federgruen and Zipkin (1984a, b), we expect an increase in the error term if we use demand distributions with larger coefficient of variation.

These observations suggest that the heuristic should perform fairly well for problems where (1) the retailer lead time is short relative to the supplier lead times, (2) the number of balanced retailers in the system is large, (3) the planning horizon is long, and (4) the demand variation is modest.

5.2. Advance Demand Information as a Substitute for Inventory

Distribution systems that incorporate advance demand information carry less inventory and are subject to lower holding and penalty costs than otherwise equivalent systems (observed from all experiments in Tables 1–6). Consider, for example, the case $L = l = 1$ in Table 5. The percentage decrease in inventory-related cost (the LB) due to advance demand information between the first experiment (in which none of the customers places orders in advance) and the sixth experiment (in which all customers during each period t place orders for period

Table 4 **Only Retailer 1 Obtains ADI** ($T = 50, h = 1, p = 19, c = 10$)

# ret	$(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$		$L=0, l=1$				$L=1, l=1$			
	ret $j \neq 1$	ret $j = 1$	$y_t(0)$	LB	UB	%	$y_t(0)$	LB	UB	%
2	(1, 0, 0, 0)	(0, 1, 0, 0)	11	2,366.61	$2{,}398.63 \pm 7.01$	1.35	10	2,327.89	$2{,}352.83 \pm 7.53$	1.07
	(2, 0, 0, 0)	(0, 2, 0, 0)	13	4,472.77	$4{,}472.77 \pm 17.72$	1.49	17	4,457.78	$4{,}549.11 \pm 14.02$	2.05
5	(2, 0, 0, 0)	(0, 2, 0, 0)	37	11,100.16	$11{,}245.02 \pm 44.14$	1.31	46	11,153.24	$11{,}396.35 \pm 34.78$	2.18
	(2, 0, 0, 0)	(0, 0, 0, 2)	32	10,903.07	$11{,}654.60 \pm 59.44$	6.89	39	10,975.78	$11{,}235.09 \pm 39.92$	2.36
	$(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$		$L=0, l=2$				$L=1, l=2$			
2	(2, 0, 0, 0, 0)	(0, 2, 0, 0, 0)	18	4,513.27	$4{,}639.86 \pm 30.13$	2.80	22	4,557.88	$4{,}619.59 \pm 14.35$	1.35
	(2, 0, 0, 0, 0)	(0, 0, 0, 0, 2)	10	4,258.37	$4{,}820.45 \pm 29.93$	13.20	13	4,311.23	$4{,}854.37 \pm 40.39$	12.60
5	(2, 0, 0, 0, 0)	(0, 2, 0, 0, 0)	48	11,352.45	$11{,}586.31 \pm 34.84$	2.06	59	11,389.77	$11{,}558.17 \pm 33.85$	1.48
	(2, 0, 0, 0, 0)	(0, 0, 0, 0, 2)	32	11,097.02	$11{,}501.29 \pm 40.86$	3.64	50	11,143.83	$11{,}548.22 \pm 48.5$	3.63

$t+3$) is 8.86%. Notice also that the system maintains lower inventory levels (zero inventory) as customers place their orders ($3 > L+l$ periods) in advance. In this sense, advance demand information is a substitute for inventory. Based on such an analysis, the inventory manager can decide whether to invest in strategies to obtain advance demand information. Notice that advance demand information allows a fundamental bridge between build-to-stock and build-to-order systems.

5.3. Advance Demand Information and Lead Time Reduction

In Figures 3(a) and (b), we plot optimal system-wide base-stock levels and the associated costs as a function of total system lead time. We systematically decrease the supplier lead time while keeping the retailers' lead time constant ($l = 2$) to illustrate the benefit of lead time reduction on total cost. Each curve represents different advance demand information scenarios. For a fixed system lead time, the difference between any two curves in Figure 3(a) and (b) gives us the reduction in inventory-related costs and base-stock levels due to advance demand information. These savings can be seen as the value of advance demand information. We also observe the reduction in base-stock levels and the costs with the system lead time. Both figures depict the joint effect of total lead time and advance demand information on the system performance.

5.4. Advance Demand Information as a Substitute for Lead Time

Consider the following three distribution systems with two retailers (from Tables 1 and 2):

ID	L, l	ADI Scenario	y^*	Cost (LB)	ADI Scenario	y^*	Cost (LB)
S1	1, 1	(2, 0, 0, 0)	20	4,510.59	(0, 2, 0, 0)	14	4,411.44
S2	0, 2	(2, 0, 0, 0, 0)	20	4,562.49	(0, 2, 0, 0, 0)	16	4,464.43
S3	1, 2	(0, 2, 0, 0, 0)	20	4,510.59	(0, 0, 2, 0, 0)	14	4,411.44

Table 5 **Cost of LB and UB for Two Retailers**

	$h=0.05, p=1, c=0.5, L=0, l=1$				$h=1, p=19, c=10, L=1, l=1$			
$(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$	$y_t(0)$	LB	UB	$(UB-LB)/LB\%$	$y_t(0)$	LB	UB	$(UB-LB)/LB\%$
(3, 0, 0, 0)	20	329.39	332.12 ± 0.07	0.83	**27**	**7,255.07**	$7{,}381.32 \pm 19.22$	1.74
(2, 1, 0, 0)	18	326.74	329.64 ± 0.07	0.89	25	7,209.71	$7{,}374.48 \pm 20.56$	2.29
(1, 1, 1, 0)	12	321.86	325.59 ± 0.07	1.16	19	7,130.77	$7{,}303.79 \pm 21.17$	2.43
(0, 1, 1, 1)	6	312.26	316.41 ± 0.06	1.33	11	6,964.17	$7{,}085.26 \pm 22.54$	1.74
(0, 0, 1, 2)	0	299.00	306.06 ± 0.06	2.36	5	6,765.43	$6{,}866.23 \pm 23.84$	1.04
(0, 0, 0, 3)	0	297.00	303.15 ± 0.06	2.07	**0**	**6,612.00**	$6{,}661.07 \pm 23.39$	0.74
						8.86%		

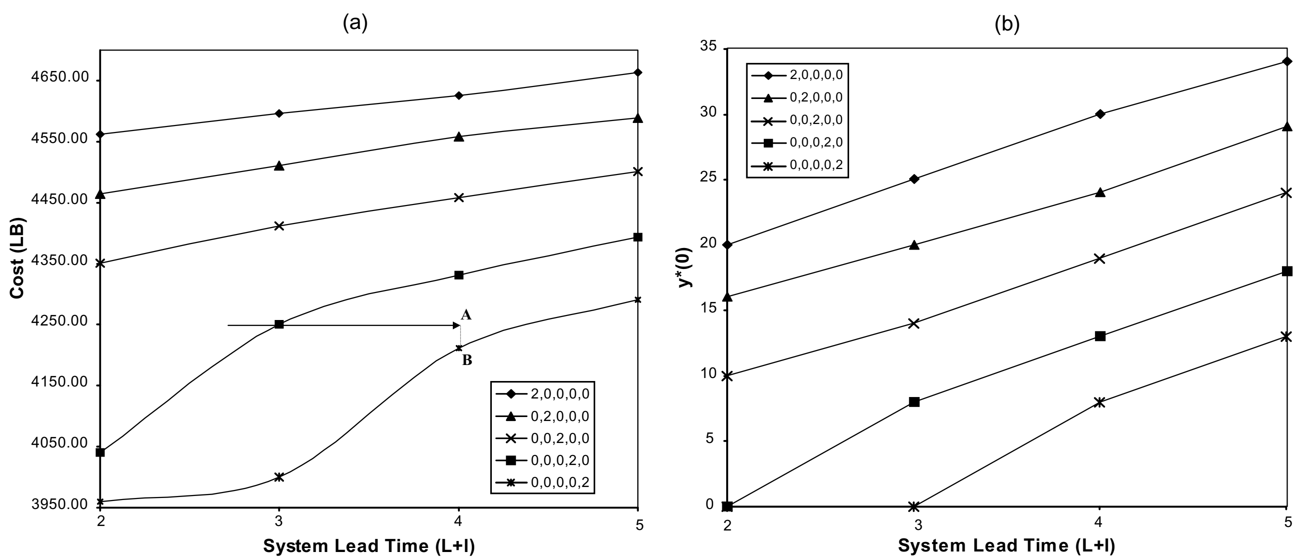
Table 6 Advance Demand Information vs. Postponement

	$y_t(0)$			Cost of LB			
	$L=0$	$L=1$	$L=2$	$L=0$	$L=1$	$L=2$	
$(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$	$l=3$	$l=2$	$l=1$	$l=3$	$l=2$	$l=1$	%
(2, 0, 0, 0)	**26**	25	**24**	**100.00**†	99.23	**98.36**	1.67
(0, 2, 0, 0)	20	20	19	97.71	96.76	95.63	2.18
(0, 0, 2, 0)	16	14	13	95.20	93.87	92.23	3.22
(0, 0, 0, 2)	**10**	8	8	**92.03**	89.86	89.13	3.25
†100.00 ≡ 5500.58				8.66	9.44	9.38	
(1, 0, 0, 0)	14	13	13	100.00‡	99.03	98.02	2.02
(0, 1, 0, 0)	10	10	10	97.36	96.30	95.01	2.47
(0, 0, 1, 0)	8	7	7	94.37	93.10	91.25	3.42
(0, 0, 0, 1)	4	4	4	90.91	88.47	87.76	3.59
‡100.00 ≡ 2809.94				9.99%	11.94%	10.46%	

Note. $N=3$, $h=1$, $c=10$, $p=19$.

Distribution systems S1 and S3 differ only in their retailer lead time and in their advance demand information scenario. Customers of S3 place orders one more period in advance than that of S1, while the *retailers* in S3 wait one period longer than the retailers in S1 to receive their orders. Observe that both cost and base-stock level for S1 are equal to those for S3. In this sense, advance demand information is a substitute for *retailer* lead time. Similarly, customers of S3 place orders one more period in advance than that of S2, which has a shorter *supplier* lead time. Notice that S3 maintains slightly lower base-stock levels and has slightly lower inventory costs than S2. System S3 accounts further gain due to the risk pooling during the first L periods. All other numerical examples also suggest that if other factors remain the same: Encouraging customers to place orders one more period in advance (1) has the same effect in holding and

Figure 3 (a) Cost vs. System Lead Time and ADI
(b) Base-Stock Levels vs. System Lead Time and ADI for ($T=50$, $h=1$, $p=19$, $c=10$)



penalty cost as reducing the *retailers*' lead time by one period, and (2) has more gains than reducing *supplier* lead time by one period. The gain due to risk pooling and advance demand information, together, outweighs the gain from reducing the supplier lead time. Another example for the later phenomenon can be observed in Figure 3(a). While B has one period longer supply lead time than A, the inventory manager in B obtains demand information one period earlier than A's manager. The difference between the costs in A and B can be attributed to the gains due to risk pooling.

For a *continuous* review *single* location inventory control problem, Hariharan and Zipkin (1995, p. 1600) show that: "Demand lead times are, in a precise sense, the opposite of supply lead times." Our numerical results suggest that this result carries over to the periodic review distribution system as an UB on the cost; that is, one would gain even more compared to the single location problems due to risk pooling.

5.5. Advance Demand Information and Postponement

We now interpret this distribution system as a multi-item production system with a common intermediate product. The warehouse represents the differentiation point. During the first L period, a common batch is produced. At the end of this period, the manager must decide how much of each item to produce and it takes l periods to produce each of these units. We address the effect of advance demand information on postponement strategies in Table 6 by altering the differentiation point L and advance demand information scenario. Columns 2–7 in Table 6 correspond to different differentiation points, whereas the rows correspond to different advance demand information scenarios. Immediate differentiation and no advance demand information is the base case for which the cost is indexed to 100. Consider, for example, the no advance demand information scenario. As we move from the column in which we have immediate differentiation $L = 0$ to the one in which we do not differentiate until the last period $L = 2$, we observe a reduction in the base-stock level from 26 to 24 and the cost from 100 to 98.36. Now, similarly consider the case of no postponement (immediate differentiation). As we move from the row in which we have no advance demand information $(2, 0, 0, 0)$ to the row in which all customers place their orders three periods in advance, we observe a reduction in the base-stock level from 26 to 10. A more complete numerical study for postponement should address the correlation among the end items. For the case of positive correlation among end items, the gains due to postponement would be lower than that of an independent case. In fact, if the products are perfectly positively correlated, there is nothing to be gained through postponement (Aviv and Federgruen 2001a). For the positive (negative) correlation case, the above numerical study provides an upper (lower) bound to the benefit of postponement. A similar numerical study can also be used to compare the benefits of obtaining advance demand information and the problem of warehouse location. Note that in the warehouse-retailer context, this section could be titled as "the joint role of advance demand information and risk pooling."

6. Restriction and Allocation

Next, we provide an explicit solution for the system-wide order-up-to level. Our aim is to gain further insights for the joint role of risk pooling and advance demand information. To obtain this closed-form solution, we assume that (1) unit holding and penalty cost is the same for all retailers, (2) $d^j_{t,s}$ follows a normal distribution with mean μ_{s-t} and variance σ^2_{s-t} for $s \in \{t, \ldots, t+N\}$, (3) allocation assumption (introduced by Eppen and Schrage 1981) holds,[10] and (4) we restrict the policy space to the class of stationary order-up-to S policies and allocate based on the solution of the myopic problem in Equation (7).

[10] Allocation Assumption. *In each allocation period t, the warehouse receives a sufficient amount so that it can bring all retailers to an equal fractile level on their respective demand distributions; that is, stock-out probability is the same at all retailers.*

Under these assumptions, the optimal order-up-to level is

$$S^* = \sum_{j=1}^{J} \sum_{s=t}^{t+L+l} o_{t,s}^{j} + J \sum_{k=0}^{L+l} (L+l+1-k)\mu_k + z.\sqrt{J\sum_{k=0}^{L}(L-k)\sigma_k^2 + J^2(l+1)\sum_{k=0}^{L}\sigma_k^2 + J^2 \sum_{k=L+1}^{L+l}(L+l+1-k)\sigma_k^2}, \tag{8}$$

where $\mu_s, \sigma_s \equiv 0$ for $s > N$ because the manager does not accept orders beyond the information horizon. We defer the derivation to Appendix B. Note first, that we can modify the term to account for the observed or the known part; that is, we define $\mathcal{S}^* \equiv S^* - \sum_{j=1}^{J}\sum_{s=t}^{t+L+l} o_{t,s}^{j}$.

We observe that the reduction in base-stock level is larger with a decrease in the mean or the variance of the demand that is closer to the current period t, other factors being equal. A reduction in μ_m (or σ_m) decreases $\mathcal{S}^*$ more than a reduction in μ_n (σ_n) for $t \le m < n \le N$. This suggests that as more of the customers place their orders in advance, the system needs a lower safety stock (eventually, the safety-stock level drops to zero if demand lead time for all customers is longer than the system lead time). Note that advance demand information reduces both the total mean and the safety stock (the last two terms in Equation (8)).

This explicit solution also sheds light on the role of risk pooling in distribution systems. For a fixed system lead time $L+l$, increasing L reduces $\mathcal{S}^*$. If we increase L to $L+1$, the term inside the square root in Equation (8) decreases by $(J^2 - J)(\sigma_0^2 + \cdots + \sigma_L^2)$ and all other terms remain the same. The gain due to risk pooling is larger for a system with several retailers that satisfy a more volatile demand. We also observe that an increase in retailer lead time l increases the uncertainty in the system more than an increase in supplier lead time L. Hence, the required safety stock is relatively larger for an increase in the retailer lead time. This discrepancy increases with the number of retailers in the system.

In conclusion, Equation (8) suggests that while advance demand information can eliminate uncertainty by asking customers to place orders in advance, the risk pooling mitigates the effect of uncertainty by pooling demand and gaining through statistical economies of scale (but never eliminating it). Given all four assumptions, Equation (8) provides a simple closed-form solution for the system-wide base-stock level. This is a nonadaptive control mechanism unlike the solution provided by the dynamic program.

7. Conclusion

In this paper, we study a distribution system with advance demand information. We show how to incorporate this new information structure and to better manage distribution systems. We propose a heuristic based on the solution of a lower bound problem. We illustrate that the heuristic, which is also used to show the benefit of advance demand information, is close to optimal. We also show that all the existing results for distribution systems under demand uncertainty apply to our case if the inventory manager is interested only in incorporating advance demand information up to retailer lead times, plus one more review period. We show that systems with advance demand information end up having lower inventory levels and inventory-related costs. In this sense, advance demand information is a substitute for inventories as well as for production and shipment lead times. Finally, we provide an explicit solution for the system-wide inventory position. We also provide some insights into the role of risk pooling and advance demand information. As pointed out by Billington and Amaral (1999), strategic evaluation of design problems requires evaluation of alternatives to identify the one that creates the greatest value. Our model provides a framework to quantify the system performance with respect to the factors discussed in this paper such as risk pooling, advance demand information, warehouse location, shipment lead times and so forth.

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Appendix A: Proofs

Proof of Theorem 1. It is based on an induction argument and is similar to Aviv and Federgruen (2001a). First, we show that the relaxation yields a dynamic programming (DP) recursion with a state space that aggregates the state variables across the retailers. We prove this first for the terminal period T. To distinguish the DP under relaxation from the original one given in Equation (3), we refer to it as J_t' and $J_{T+1}' \equiv 0$. Note that for $t = T$, o_t is a vector of zeros because we do not accept orders beyond $T+l$ by assumption.

$$\begin{aligned}
J_T'(x_T, v_T, o_T) &= \min_{\{w_T,\, y_T^j : \sum_{j=1}^J (y_T^j - x_T^j) = w_{T-L},\, w_T \ge 0\}} \left\{ c_T^0 w_T + \sum_{j=1}^J G_T^j(y_T^j) \right\} \\
&= \min_{\{w_T,\, Y_T : Y_T = \sum_j x_T^j + w_{T-L},\, w_T \ge 0\}} \Bigg\{ c_T^0 w_T \\
&\qquad + \min_{\{y_T^j : \sum_j y_T^j = Y_T\}} \sum_j G_T^j(y_T^j) \Bigg\} \\
&= \min_{\{w_T,\, Y_T : Y_T = \sum_j x_T^j + w_{t-L},\, w_t \ge 0\}} \left\{ c_T^0 w_T + R_T(Y_T) \right\} \\
&= V_T\left(\sum_j x_T^j, v_T, O_T \right).
\end{aligned}$$

Now, assume that the theorem is true by induction for $t+1$, and we have $V_{t+1}(X_{t+1}, v_{t+1}, O_{t+1}) = J_{t+1}'(x_{t+1}, v_{t+1}, o_{t+1})$. So, we have

$$\begin{aligned}
&J_t'(x_t, v_t, o_t) \\
&= \min_{\{w_t,\, y_t^j : \sum_{j=1}^J (y_t^j - x_t^j) = w_{t-L},\, w_t \ge 0\}} \Bigg\{ c_t^0 w_t + \sum_j G_t^j(y_t^j) \\
&\qquad + \alpha E V_{t+1}\left(\sum_j y_t^j - \sum_{s=t}^{t+l+1} D_{t,s} - O_{t,t+l+1}, O_{t+1} \right) \Bigg\} \\
&= \min_{\{w_t,\, Y_t : Y_t = \sum_j x_t^j + w_{t-L},\, w_t \ge 0\}} \Bigg\{ c_t^0 w_t + \min_{\{y_t^j : \sum_j y_t^j = Y_t\}} \left\{ \sum_j G_t^j(y_t^j) \right\} \\
&\qquad + \alpha E V_{t+1}(X_{t+1}, v_{t+1}, O_{t+1}) \Bigg\} \\
&= \min_{\{w_t,\, Y_t : Y_t = \sum_j x_t^j + w_{t-L},\, w_t \ge 0\}} \left\{ c_t^0 w_t + R_t(Y_t) + \alpha E V_{t+1}(X_{t+1}, v_{t+1}, O_{t+1}) \right\} \\
&= V_t\left(\sum_j x_t^j, v_t, O_t \right).
\end{aligned}$$

This proves the equivalence of J_t' (the DP with relaxation) and V_t (the DP with the aggregate state space). Note that because we relax a constraint, the action space is larger than the previous case. This implies $J_t'(x_t, v_t, o_t) \le J_t(x_t, v_t, o_t)$, which concludes the first part of the theorem. R_t inherits the convexity property simply from $G_t^j(y_t^j)$, concluding the proof. □

Proof of Theorem 2. By induction on t, we prove first Parts 1–3. For $t = T$, $H_T(y, O_T) = c_T^0 y + R_T(y)$, which is convex in y for any given vector O_T due to Theorem 1. Assume by induction that Part 1 is true for $t+1$. Let $y_{t+1}^*(O_{t+1})$ be the smallest y that minimizes $H_{t+1}(\cdot, O_{t+1})$. This implies the second part of the theorem; if $x \le y_{t+1}^*(O_{t+1})$, then order up to $y_{t+1}^*(O_{t+1})$. Optimal value $V_{t+1}(x, O_{t+1}) + c_{0t+1}x = H_{t+1}(\max(x, y_{t+1}^*(O_{t+1})), O_{t+1})$. This function is increasing convex in x because $H_{t+1}(x, O_{t+1})$ is convex and increasing for $x > y_{t+1}^*(O_{t+1})$ due to our induction hypothesis. To complete the induction argument for Parts 1–3, it suffices to show that Part 1 is true for t. This follows from the definition of H_t and noticing that (1) the update X_{t+1} is linear in x, (2) the composition of increasing convex function and a linear function is convex, (3) expectation (and summation) preserves convexity, and (4) $R_t(\cdot)$ is convex.

Before addressing Parts 4–6, we prove a preliminary result. We define $\nabla h(x, y) = h(x+1, y) - h(x, y)$ as the first difference of function $h(x, y)$.

Remark 1. If $f(x)$ and $g(x)$ are two convex functions and x_f, x_g are the smallest minimizer of $f(\cdot)$ and $g(\cdot)$, respectively, and $\nabla f(x) \ge \nabla g(x)$, then $x_f \le x_g$.

Proof. Assume for a contradiction that $x_f > x_g$, we have $\nabla g(x_g - 1) < 0 \le \nabla g(x_f - 1) \le \nabla f(x_f - 1)$. The first inequality is due to the optimality of x_g. The second one is due to the assumption $x_f > x_g$. The third one is due to the statement $\nabla f(x) \ge \nabla g(x)$. Then, we have $\nabla f(x_f - 1) > 0$ but this contradicts the optimality of x_f. Hence, $x_f \le x_g$.

First, let us prove that $y_t^*(O_t + \epsilon e_1) - y_t^*(O_t) \le \epsilon$, that is Part 5 holds for $j = 1$ and *for all* t. We have for any x

$$\begin{aligned}
\nabla H_t(x, O_t + \epsilon e_1) &= c_t^0 + \nabla R_t(x) \\
&\quad + \alpha E \nabla V_{t+1}\left(x - O_{t,t+l+1} - \epsilon - \sum_{s=t}^{t+l+1} D_{t,s}, O_{t+1} \right) \\
&\ge c_t^0 + \nabla R_t(x - \epsilon) \\
&\quad + \alpha E \nabla V_{t+1}\left(x - O_{t,t+l+1} - \epsilon - \sum_{s=t}^{t+l+1} D_{t,s}, O_{t+1} \right) \\
&= \nabla H_t(x - \epsilon, O_t).
\end{aligned} \tag{9}$$

The inequality is due to the convexity of $R_t(x)$. Note that the smallest minimizer of $H_t(x - \epsilon, O_t)$ is nothing but $y_t^*(O_t) + \epsilon$. Also $y_t^*(O_t + \epsilon e_1)$ is the smallest minimizer of $H_t(\cdot, O_t + \epsilon e_1)$, which is convex from Part 1. From inequality (9) and Remark 1, we conclude that $y_t^*(O_t + \epsilon e_1) \le y_t^*(O_t) + \epsilon$. Note that

$$V_t(x, O_t) = -c_t^0 x + \begin{cases} H_t(y_t^*(O_t), O_t), & x_t < y_t^*(O_t), \\ H_t(x, O_t), & x \ge y_t^*(O_t). \end{cases} \tag{10}$$

Next, we prove $\nabla V_t(x - \epsilon, O_t) \le \nabla V_t(x, O_t + \epsilon e_1)$ for all t. To do this, we divide the domain of these two functions into four mutually exclusive, collectively exhaustive cases.

Case 1. If $x - \epsilon < y_t^*(O_t)$ and $x < y_t^*(O_t + \epsilon e_1)$, then $\nabla V_t(x - \epsilon, O_t) = -c_t^0 = \nabla V_t(x, O_t + \epsilon e_1)$.

Case 2. If $x - \epsilon < y_t^*(O_t)$ and $x \ge y_t^*(O_t + \epsilon e_1)$, then $\nabla V_t(x - \epsilon, O_t) = -c_t^0 \le -c_t^0 + \nabla H_t(x, O_t + \epsilon e_1) = \nabla V_t(x, O_t + \epsilon e_1)$. The inequality is due to the fact that $H_t(\cdot, O_t + \epsilon e_1)$ is increasing for $x \ge y_t^*(O_t + \epsilon e_1)$ (which is due to Parts 1 and 2).

Case 3. The case $x - \epsilon \ge y_t^*(O_t)$ and $x < y_t^*(O_t + \epsilon e_1)$ is not possible because we already showed that $y_t^*(O_t + \epsilon e_1) \le y_t^*(O_t) + \epsilon$.

Case 4. If $x - \epsilon \ge y_t^*(O_t)$ and $x \ge y_t^*(O_t + \epsilon e_1)$, then $\nabla V_t(x - \epsilon, O_t) = -c_t^0 + \nabla H_t(x - \epsilon, O_t) \le -c_t^0 + \nabla H_t(x, O_t + \epsilon e_1) = \nabla V_t(x, O_t + \epsilon e_1)$. The inequality is due to Equation (9).

All four cases show that $\nabla V_t(x-\epsilon, O_t) \le \nabla V_t(x, O_t+\epsilon e_1)$ for all t. From the definition of H_t, the update X_{t+1}, and the above cases we conclude that $\nabla H_t(x, O_t+\epsilon e_1) = -c_t^0 + \nabla R_t(x) + \alpha E\nabla V_{t+1}(x - O_{t,t+l+1} - \epsilon - \sum_{s=t}^{t+l+1} D_{t,s}, O_{t+1}) \le -c_t^0 + \nabla R_t(x) + \alpha E\nabla V_{t+1}(x - O_{t,t+l+1} - \sum_{s=t}^{t+l+1} D_{t,s}, O_{t+1}+\epsilon e_1) = \nabla H_t(x, O_t+\epsilon e_2)$.

Next, we want to show by induction that

$$\nabla H_t(x, O_t+\epsilon e_j) \le \nabla H_t(x, O_t+\epsilon e_{j+1}) \tag{11}$$

is true also for all $j > 1$. Now, assume for an induction argument that it is true for an arbitrary j, then Part 1 and Remark 1 imply that Part 4 for j is true. To prove Equation (11) for $j+1$, we first prove $\nabla V_t(x, O_t+\epsilon e_j) \le \nabla V_t(x, O_t+\epsilon e_{j+1})$. For this, we consider only three cases because we already showed that $y_t^*(O_t+\epsilon e_j) \ge y_t^*(O_t+\epsilon e_{j+1})$ and use Equation (10).

Case 1. If $x \ge y_t^*(O_t+\epsilon e_j)$, then $\nabla V_t(x, O_t+\epsilon e_j) = -c_t^0 + \nabla H_t(x, O_t+\epsilon e_j) \le -c_t^0 + \nabla H_t(x, O_t+\epsilon e_{j+1}) = \nabla V_t(x, O_t+\epsilon e_{j+1})$.

Case 2. If $y_t^*(O_t+\epsilon e_j) > x \ge y_t^*(O_t+\epsilon e_{j+1})$, then $\nabla V_t(x, O_t+\epsilon e_j) = -c_t^0 \le -c_t^0 + \nabla H_t(x, O_t+\epsilon e_{j+1}) = \nabla V_t(x, O_t+\epsilon e_{j+1})$ because $H_t(x, O)$ is increasing for $x \ge y_t^*(O)$.

Case 3. If $y_t^*(O_t+\epsilon e_{j+1}) > x$, then $\nabla V_t(x, O_t+\epsilon e_j) = -c_t^0 = \nabla V_t(x, O_t+\epsilon e_{j+1})$.

The above three cases prove that $\nabla V_t(x, O_t+\epsilon e_j) \le \nabla V_t(x, O_t+\epsilon e_{j+1})$. From the definition of H_t, we obtain $\nabla H_t(x, O_t+\epsilon e_{j+1}) = -c_t^0 + \nabla R_t(x) + E\nabla V_{t+1}(x - O_{t,t+l+1} - \sum_{s=t}^{t+l+1} D_{t,s}, O_{t+1}+\epsilon e_j) \le -c_t^0 + \nabla R_t(x) + E\nabla V_{t+1}(x - O_{t,t+l+1} - \sum_{s=t}^{t+l+1} D_{t,s}, O_{t+1}+\epsilon e_{j+1}) = \nabla H_t(x, O_t+\epsilon e_{j+2})$. Note that due to the state variable update, the perturbation in the observed demand for ϵe_{j+1} in period t becomes a perturbation of ϵe_j in period $t+1$. This completes the induction argument for Equation (11) and the proof of Part 4. Recall that Part 5 for $j=1$ was proved earlier. Hence, the result Part 5 for $j>1$ immediately follows from Part 4 since $y_t^*(O_t+\epsilon e_j) - y_t^*(O_t) \le y_t^*(O_t+\epsilon e_{j-1}) - y_t^*(O_t) \le \cdots \le y_t^*(O_t+\epsilon e_1) - y_t^*(O_t) \le \epsilon$. The proof of Part 6 is similar to the Proof of Theorem 4, Part 5 in Gallego and Özer (2001).

Proof of Lemma 1. The difference between $\tilde{V}_t$ and V_t consists of the costs during the periods $\{t, \ldots, t+L\}$. Note that the orders initiated at the beginning of period t will not arrive until the beginning of period $t+L$. Hence, the cost incurred between these periods is given by $r_t(X_t, v_t, O_t) = R_t(X_t + w_{t-L}) + \alpha E R_{t+1}(X_t + w_{t-L} + w_{t-L+1} - \sum_{s=t}^{t+l+1} D_{t,s} - O_{t,t+l+1}) + \cdots + \alpha^{L-1} E R_{t+L-1}(X_t^\Delta - \sum_{r=t}^{t+L-2}\sum_{s=r}^{r+l+1} D_{r,s} - \sum_{s=t}^{t+L-2} O_{s,s+l+1})$. This function is clearly independent of the decision variables w_t and y_t. □

Appendix B: Derivation of S^*

Under all four assumptions in §6, at the beginning of period t, the inventory manager monitors the system-wide inventory position and orders to bring this level up to S. Orders from the outside supplier arrive L periods later. The inventory to be allocated at time $t+L$ is then given by $S-V$, where $V = \sum_{j=1}^{J}\sum_{s=t}^{t+L-1}(o_{t,s}^j + u_{t,s}^j)$. Given the above assumptions, the expected costs charged to period $t+L$ would be the same as period $t+L+1$ for all t. Hence, we focus only on the single-period cost charged to period $t+L$. That is,

$$\min_{\tilde{y}_{t+L}^j\ \forall j}\left\{\sum_{\forall j} E[h(\tilde{y}_{t+L}^j - W^j)^+ + p(\tilde{y}_{t+L}^j - W^j)^-]\quad \text{subject to } \sum_{\forall j}\tilde{y}_{t+L}^j = S - V\right\},$$

where $\tilde{y}^j$ is the classical inventory position (that is, it is not yet modified) and $W^j = \sum_{s=t+L}^{t+L+l}(o_{t,s}^j + u_{t,s}^j)$. As in Zipkin (1982), introducing the Lagrange multiplier λ and solving for the minimizers of the Lagrangian function, we obtain the optimal allocation for each retailer; that is,

$${}^*y_{t+L}^j = EW^j + \sqrt{\mathrm{Var}[W^j]}\,\Phi^{-1}\left(\frac{p-\lambda}{p+h}\right),$$

where $\Phi(\cdot)$ is the cdf of standard normal. Note that from our assumption (2), we have $EW^j = \sum_{s=t+L}^{t+L+l} o_{t,s} + (l+1)\sum_{k=0}^{L}\mu_k + \sum_{k=L+1}^{L+l}(L+l+1-k)\mu_k$ and $\mathrm{Var}[W^j] = (l+1)\sum_{k=0}^{L}\sigma_k^2 + \sum_{k=L+1}^{L+l}(L+l+1-k)\sigma_k^2$. We sum ${}^*y_{t+L}^j$ across all retailers and equate it to $S-V$ and solve for $\Phi^{-1}(p-\lambda)/(p+h)$. By substituting this expression, we obtain

$${}^*y_{t+L}^j = \sum_{s=t+L}^{t+L+l} o_{t,s}^j + \frac{S - V - \sum_{j=1}^{J}\sum_{s=t+L}^{t+L+l} o_{t,s}^j}{J}.$$

If the inventory position at each retailer brought to ${}^*y_{t+L}^j$, then the stock at the end of period $t+L+l$ will be a random quantity and is given by

$$\begin{aligned}{}^*y_{t+L}^j - W^j &= \left[\frac{S - \sum_{j=1}^{J}\sum_{s=t+L}^{t+L+l} o_{t,s}^j}{J}\right] - \left[\sum_{s=t+L}^{t+L+l} u_{t,s}^j + \frac{1}{J}\sum_{i=1}^{J}\sum_{s=t}^{t+L-1} u_{t,s}^i\right]\\ &= s^j - \xi^j.\end{aligned}$$

The expected single-period cost for each retailer j is $G^j(S) = hE(s^j - \xi^j)^+ + pE(s^j - \xi^j)^-$, which is a function of S and the expectation is with respect to ξ^j. Note that the minimizer of this function is achieved by choosing S such that $P(\xi^j \le s^j) = p/(p+h)$. From this, we have

$$z = \Phi^{-1}\left(\frac{p}{p+h}\right) = \frac{s^j - E\xi^j}{\sqrt{\mathrm{Var}[\xi^j]}},$$

where $E\xi^j = \sum_{k=0}^{L+l}(L+l+1-k)\mu_k$ and $\mathrm{Var}[\xi^j] = (l+1)\sum_{k=0}^{L}\sigma_k^2 + \sum_{k=L+1}^{L+l}(L+l+1-k)\sigma_k^2 + \frac{1}{J}\sum_{k=1}^{L}(L-k)\sigma_k^2$. Solving for S, we obtain the explicit solution in Equation (8), which is independent of j. This means that while it is a minimizer of $G^j(\cdot)$, it is also a minimizer of $\sum_{\forall j} G^j(\cdot)$. It is, therefore, the optimal system-wide order-up-to level. □

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