

# Promised Lead-Time Contracts Under Asymmetric Information

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We study the important problem of how a supplier should optimally share the consequences of demand uncertainty (i.e., the cost of inventory excesses and shortages) with a retailer in a two-level supply chain facing a finite planning horizon. In particular, we characterize a multiperiod contract form, the *promised lead-time contract*, that reduces the supplier's risk from demand uncertainty and the retailer's risk from uncertain inventory availability. Under the contract terms, the supplier guarantees on-time delivery of complete orders of any size after the promised lead time. We characterize the optimal promised lead time and the corresponding payments that the supplier should offer to minimize her expected inventory cost, while ensuring the retailer's participation. In such a supply chain, the retailer often holds private information about his shortage cost (or his service level to end customers). Hence, to understand the impact of the promised lead-time contract on the supplier's and the retailer's performance, we study the system under *local control with full information* and *local control with asymmetric information*. By comparing the results under these information scenarios to those under a *centrally controlled system*, we provide insights into stock positioning and inventory risk sharing. We quantify, for example, how much and when the supplier and the retailer overinvest in inventory as compared to the centrally controlled supply chain. We show that the supplier faces more inventory risk when the retailer has private service-level information. We also show that a supplier located closer to the retailer is affected less by information asymmetry. Next, we characterize when the supplier should optimally choose not to sign a promised lead-time contract and consider doing business under other settings. In particular, we establish the optimality of a *cutoff level policy*. Finally, under both full and asymmetric service-level information, we characterize conditions when optimal promised lead times take extreme values of the feasible set, yielding the supplier to assume all or none of the inventory risk—hence the name *all-or-nothing solution*. We conclude with numerical examples demonstrating our results.

*Subject classifications:* inventory/production: multi-item/echelon/stage; uncertainty: stochastic; operating characteristics; game theory: mechanism design; finite types; screening.

*Area of review:* Manufacturing, Service, and Supply Chain Operations.

*History:* Received April 2004; revisions received November 2005, August 2006, January 2007; accepted January 2007.

## 1. Introduction

Consider a supply chain with a supplier and a retailer. Both firms face positive replenishment lead times. Inventories at both locations are managed periodically over a finite planning horizon. Customer demand is satisfied only through the retailer. The objective of each firm is to minimize the ordering, holding, and shortage costs over the planning horizon due to demand and supply uncertainty. In such a supply chain, the supplier prefers the retailer to place orders well in advance of his requirement. However, the retailer prefers the supplier to fulfill orders immediately without facing any backlog at the supplier's site. Hence, the supply chain faces an incentive problem in which both the supplier and the retailer want the other party to bear the consequences of demand uncertainty. How should this supply chain's inventory be managed?

To address the need for proper sharing of expected inventory cost, we consider a *promised lead-time contract*. Under this contract, the retailer places advance orders with the supplier. The supplier guarantees shipment of each order

on time and in full after a *promised lead time*. A promised lead-time contract eliminates the retailer's risk from uncertain supply, but extends the retailer's forecast horizon beyond his standard replenishment lead time. This contract also provides the supplier with advance orders, thereby decreasing her risk from uncertain demand. A cost-benefit analysis of this interaction, and the resulting inventory costs, determine who pays for the promised lead-time agreement.

Single-sourcing relationships rely on issues such as contingency planning, inventory, and on-time delivery, so the study of promised lead-time contracts is both relevant and timely (Hauser 2003). Billington (2002) and Cohen et al. (2003) report that among the most common forms of contracts are those specifying lead times. Also, because planning systems such as MRP (materials resource planning) stipulate a delivery lead time, purchasing agents are often driven by a delivery lead time when placing orders. These business processes require firms to agree on delivery lead times—hence the name promised lead time.

In this paper, we characterize the optimal promised lead time and the corresponding payment that the supplier should offer to minimize her expected inventory cost while ensuring the retailer's participation. To do so, we first characterize both firms' optimal stocking and replenishment decisions under a given promised lead-time contract. To understand the impact of the promised lead-time contract on the supplier's and the retailer's performance, we study the supply chain under three control mechanisms: central control, local control with full information, and local control with asymmetric information. By comparing the results under these different control and information scenarios, we also provide insights into stock positioning and inventory risk sharing.

Under *central control*, the two-stage system's inventory and information is managed by a single decision maker. The manager optimally allocates inventory to minimize the overall expected inventory cost. Hence, this system does not face the aforementioned incentive problem. Clark and Scarf's (1960) seminal paper addresses how to optimally manage this serial system. In this paper, we provide closed-form solutions for allocating inventory to each stage. These solutions are new and provide insights into stock positioning through simple spreadsheet calculations. They also serve as a comparison point for a supply chain under local control, which is the main focus of this paper.

Under *local control with full information*, each firm locally controls its own inventory. The supplier offers the promised lead-time contract while having full information about the retailer's inventory-related costs. We show that the optimal promised lead time is decreasing in the retailer's shortage cost or, equivalently, his service level to end customers. In other words, the supplier optimally faces larger inventory cost when working with a retailer who provides high service to customers. We show the conditions under which the supplier's choice of promised lead time generates the supply chain optimal promised lead-time contract. We also characterize the optimal stocking levels under this regime. By comparing these results to the central control case, we quantify *how much* and *when* the supplier and the retailer over- or underinvest in inventory.

Under local control, another incentive problem may arise. Specifically, the retailer can influence the supplier's contract design by exaggerating the severity of his inventory-related costs from demand uncertainty. The inventory holding costs are relatively easier to assess because they are mainly based on factors observable to the public. However, companies often state shortage cost as a strategic cost never to be revealed to competitors and suppliers. During our interactions with a telecommunication equipment distributor and a data storage technology provider, we observed that these two firms employ different accounting formulas that describe shortage cost. The inputs to these formulas are mainly classified under two categories of data: market specific and company specific. The market-specific factors influencing a retailer's unit shortage cost

consist of factors such as customer patience and the amount of goodwill loss. The confidential company-specific factors stem from the opinions of executives regarding the company's competitive strategy, market position, and assessment of inventory risks. These factors are not observable by other firms. The retailer translates both market-specific and company-specific factors into shortage cost by using a formula that is also unobservable to the other firms. Hence, the retailer often keeps private information regarding his shortage cost. We refer to this case as *asymmetric shortage cost information*. Throughout the paper, we use service level and shortage cost interchangeably because in our setting a service level implies a shortage cost, and vice versa.

Under the aforementioned asymmetric information case, the supplier may first try to obtain the cost information, either by asking the retailer or by observing the retailer's replenishment orders. However, we show that the retailer has an incentive to exaggerate his service level when asked for this information. We also show that the supplier cannot learn the retailer's service level by observing retailer orders over time. Hence, *credible* information sharing is not possible without a proper incentive mechanism. In this paper, we structure a contract mechanism that the supplier can offer to obtain credible service-level information while minimizing her inventory-related cost over the planning horizon. Under this mechanism, the supplier offers a menu of promised lead times with corresponding payments to the retailer. The retailer voluntarily chooses a promised lead time  $\tau$  and pays  $K(\tau)$  each period to guarantee contract terms. We also compare the resulting inventory costs and allocation under this mechanism to those under the full-information scenario. This comparison provides insights into the value of knowing the retailer's service level through the use of technologies such as RFID (radio frequency identification device) and compliance management systems.<sup>1</sup> We also compare the resulting inventory allocation to those under central control.

Next, we characterize when the supplier should optimally forgo establishing a supply chain by inducing the retailer to sign a promised lead-time contract. This situation arises when the supplier is not willing to incur an inventory cost that is larger than her reservation cost. In particular, we show that the supplier optimally induces the retailer to sign the promised lead-time contract only when the retailer's service level to end customers is lower than a *cutoff level*. We also show that the cutoff level is increasing in both the supplier's and the retailer's reservation costs. In addition, we characterize the optimality of the cutoff policy. Finally, we characterize the conditions under which optimal promised lead times under both full and asymmetric service-level information take extreme values of the feasible set, yielding the supplier to assume all or none of the inventory risk; i.e., an *all-or-nothing solution*. We conclude with numerical examples demonstrating our results.

The remainder of this paper is organized as follows. In §2, we review related research. In §3, we explicitly

describe the supply chain model and the promised lead-time contract. In §4, we formulate the problem under central control. In §5, we study the supplier's and the retailer's inventory decisions under a promised lead-time contract. We analyze the supplier's optimal promised lead-time contract under local control with full and asymmetric service-level information. In §6, we compare the resulting inventory levels under local control to those under central control. In §7, we study a special case: an all-or-nothing type inventory risk-sharing agreement. In §8, we provide numerical examples. Finally, in §9, we conclude.

## 2. Literature Review

Effective inventory management and sharing the consequences of demand uncertainty among supply chain members over a *planning horizon* is one of the most fundamental and important problems in operations management. However, the literature on contracting to share inventory risk mainly considers a single-period interaction in a two-level supply chain (Cachon 2003). To the best of our knowledge, Cachon and Zipkin (1999) is the first paper that provides contract mechanisms designed for two firms facing a long inventory-planning horizon. In this respect, our paper is related to theirs. However, we address a finite-horizon problem under information asymmetry and consider a different contract mechanism. They study an average cost criterion under full information, but they also consider competition. In their paper, the retailer faces shortages due to supply restrictions at the supplier. The supplier pays a fraction of the penalty cost for shortages to the retailer. In our paper, the retailer is guaranteed ample supply. Essentially, the contract mechanism, information, and supply chain structures studied in our and their papers are different. Lee and Whang (2000) and Porteus (2000) also study incentive issues in a serial supply chain. Lee and Whang discuss how Clark and Scarf's (1960) optimal replenishment policy for a centrally managed serial system can be implemented by managers located in each stage. Porteus (2000) provides a way of implementing the scheme in Lee and Whang (2000). Unlike our paper, both of these papers assume that a central decision maker first determines the optimal inventory control policy. They also assume that inventories and cost information at each location are monitored and shared among all managers in the supply chain.

Full information, central control, and monitoring are strong assumptions. Often, supply chain partners hold private information. The informed partner may use this information to improve profits, at the expense of the other partner. The less-informed partner can solve an adverse selection, or screening problem, to devise an optimal contract mechanism that minimizes information rent. Examples of adverse selection problems in the operations literature can be found in Cachon and Lariviere (2001), Corbett (2001), Ha (2001), Özer and Wei (2006), and Cachon and Zhang (2006). For a comprehensive survey, we refer the

reader to Chen (2003). Within this stream of literature, the closest work to ours is the work of Corbett (2001). The author considers optimal contracts between a supplier and a retailer following a  $(Q, r)$  inventory control policy to minimize average cost. The supplier chooses the batch size  $Q$ ; the retailer determines the reorder point  $r$ . When the retailer has private information about his backorder penalty cost, the supplier offers an optimal contract mechanism while screening the retailer's type. Similar to Corbett (2001), we study the impact of information asymmetry regarding the backorder penalty cost.

Most supply chain papers on screening implicitly assume that the principal wants to induce the agent to establish a supply chain by agreeing to the contract terms. In this paper, we also consider the supplier's (the principal's) participation constraint explicitly as in Corbett and de Groote (2000) and Ha (2001). These authors characterize cutoff level policies. As in Ha (2001), we also characterize the optimality of the cutoff policy. Incidentally, management consultants also identify the need to segment retailers with differing service-level requirements and to adapt supply chain policies to serve each segment with maximum profitability (Anderson et al. 1997, Gadiesh and Gilbert 1998). Cohen et al. (2003) address this problem in a service parts supply chain.

Finally, a group of researchers explores delivery lead-time commitments and quotations under nonstrategic settings. For example, Barnes-Schuster et al. (2006) study a two-level supply chain under full information and normally distributed demand. They provide conditions under which the retailer or supplier should hold the entire inventory. This result is related to the one in §7 for full information. See also Keskinocak et al. (2001), Wang et al. (2002), and the references therein.

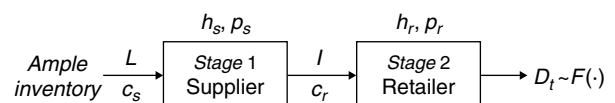
## 3. The Model

Here we describe three important aspects of the model: the two-stage supply chain, the promised lead-time contract, and the control mechanism under different information scenarios.

### 3.1. Two-Stage Supply Chain

We study the two-stage supply chain shown in Figure 1. Uncertain end-customer demand is satisfied through the retailer. Demand  $D_t$  in each period  $t$  is modeled by a sequence of nonnegative, independent and identically distributed (i.i.d.) random variables drawn from a stationary distribution with c.d.f.  $F(\cdot)$ , density  $f(\cdot)$ , and finite mean

**Figure 1.** Two-stage supply chain.



$\mu \geq 0$ . The supplier and the retailer both know the demand distribution.<sup>2</sup> We denote demand over the current and the next  $n$  periods as<sup>3</sup>  $D^{n+1} \equiv \sum_{k=1}^{l+n} D_k$ .

The inventory at each stage is reviewed periodically. The sequence of events is as follows. Both the supplier and the retailer receive their respective shipments at the beginning of a period. The supplier orders from an upstream firm with ample inventory and will receive these orders  $L$  periods later. Shipments from the supplier to the retailer take  $l$  periods to arrive. The supplier and retailer incur linear cost  $c_s > 0$  and  $c_r > 0$ , respectively, per item ordered, and no fixed cost for placing an order. At the end of the review period, customer demand is realized. The retailer satisfies demand through on-hand inventory. Unsatisfied demand is backlogged. Backorders of end-customer demand incur a unit penalty cost  $p_r$  per period only at the retailer. The supplier incurs either an explicit or an implicit shortage cost based on the control structure we specify later. The supplier and the retailer incur unit installation holding cost  $h_s > 0$  and  $h_r > 0$ , respectively, where  $h_s \leq h_r$  for any inventory remaining at the end of each period. At the end of period  $T$ , leftover inventory (respectively, backlog) is salvaged (respectively, purchased) at a linear per-unit value of  $c_s$  and  $c_r$  at each stage, respectively. This sequence and set of assumptions are the classical ones portrayed, for example, in Veinott (1965).

### 3.2. Promised Lead-Time Contract

A *promised lead-time contract* has two parameters: promised lead time  $\tau$  and corresponding per-period lump-sum payment  $K$ . Under this contract, the retailer places each of his orders  $\tau$  periods in advance of his needs. The supplier guarantees to ship this order, in full, after  $\tau$  periods. The retailer must wait another  $l$  periods (shipping time) before receiving the guaranteed order. To do so, the supplier arranges an alternate sourcing strategy to fill any retailer demand that exceeds the supplier's on-hand inventory. The supplier borrows emergency units from this alternative source and incurs penalty  $p_s$  per unit per period until the alternative source is replenished.<sup>4</sup> Under this contract, the supplier is responsible for the supply risk for a finite planning horizon. The analysis of this interaction and the resulting inventory cost determine the lump-sum payment between the firms. When  $K$  is positive, we interpret this transaction as a payment from the retailer to the supplier.

The effect of a promised lead time is to shift inventory-related costs due to demand uncertainty from the supplier to the retailer. With a promised lead time, the supplier learns the retailer's order  $\tau$  periods in advance. On one hand, for the supplier, a promised lead time changes the number of periods of demand uncertainty from  $L + 1$  to  $L + 1 - \tau$ . On the other hand, for the retailer, a promised lead time increases the number of periods of demand uncertainty from  $l + 1$  to  $l + 1 + \tau$ . Note that when  $\tau = 0$ , the retailer demands immediate shipment of all orders, and the supplier's operation is completely build-to-stock.

When  $\tau = L + 1$ , exceeding the supplier's replenishment lead time, the supplier builds to order for the retailer and does not need to carry any inventory. Hence, any reasonable promised lead time would satisfy  $\tau \in \{0, \dots, L + 1\}$ .

### 3.3. System Control

Our base case is the system under *central control* without a promised lead-time contract. For this case, a central decision maker, who has full information about the firms' operations, sets the inventory-ordering policy for both the supplier and the retailer. In particular, she decides how to allocate inventory within the system so as to minimize the total expected inventory cost. The supplier satisfies the retailer's order from inventory on hand at the supplier's location. Any unsatisfied order is backlogged. Hence, the retailer faces supply risk. This inventory control problem is the classical Clark and Scarf (1960) model discussed in §4.

Our main case is the system under *local control*. This system consists of two distinct decision makers: the supplier and the retailer. Each firm's goal is to minimize its own expected inventory cost by placing a replenishment order based on its on-hand and pipeline inventory and backlog levels—hence the name local control. Under the *full-information scenario*—i.e., when the supplier knows the retailer's inventory ordering, holding, and shortage costs—the supplier designs a promised lead-time contract and offers it to the retailer. If the retailer accepts the contract, both firms are committed to the promised lead-time contract terms for a finite horizon. The retailer, however, often has private information about his end-customer service level. Under this *asymmetric information scenario*, the supplier offers a menu of contracts that minimizes her expected inventory cost. The retailer chooses a promised lead-time contract that minimizes his expected inventory cost. After the contracting stage, under both full and symmetric information, the sequence of events is as described in §3.1, and each firm independently decides how to replenish its local on-hand inventory to minimize its own expected inventory cost. Note that a third-party decision maker who has full information may also choose the optimal promised lead-time contract for the entire supply chain, while the supplier and the retailer set their own stocking levels. This third-party decision maker's contract choice is what will be referred to as the *first-best contract under local control*.<sup>5</sup> We address the local control system in §5.

Under both control systems, each firm optimally follows a base-stock policy as discussed next. Hence, we describe the replenishment decisions as choosing stocking levels.

## 4. Central Control and the System-Optimal Solution

The system-optimal solution minimizes the total expected, discounted inventory cost for a finite-horizon, periodic-review inventory control problem in series. Clark and Scarf (1960) show that an echelon base-stock policy is

optimal for this classical serial system. Other papers provide simpler proofs, efficient computational methods, and extensions to the infinite-horizon case (see, for example, Federgruen and Zipkin 1984, Chen and Zheng 1994). For a stationary serial system, Gallego and Özer (2003) show that a myopic base-stock policy is optimal. They show how to allocate costs to firms (stages) in a certain way to obtain optimal base-stock levels. Next, we summarize this method. Let  $x_{jt}^e$  and  $y_{jt}^e$  be firm  $j$ 's echelon inventory position<sup>6</sup> before and after ordering, respectively, in period  $t$ , where  $j = s$  represents the supplier and  $j = r$  the retailer. Also, let

$$\mathcal{L}_r(y_{rt}^e) = (1 - \alpha)c_r y_{rt}^e + \alpha^l E[(h_r - h_s)(y_{rt}^e - D^{l+1}) + (p_r + h_r)(y_{rt}^e - D^{l+1})^-] \quad (1)$$

be the retailer's expected inventory cost in period  $t$  where  $\alpha$  is the discount factor. Define  $y_r^m$  as the smallest minimizer of  $\mathcal{L}_r(\cdot)$ . This minimizer is the retailer's optimal base-stock level. Next, define the implicit penalty cost

$$IP^m(y) = \mathcal{L}_r(\min\{y_r^m, y\}) - \mathcal{L}_r(y_r^m). \quad (2)$$

The supplier's expected cost charged in period  $t$  is then

$$\mathcal{L}_s(y_{st}^e) = (1 - \alpha)c_s y_{st}^e + \alpha^L E[h_s(y_{st}^e - D^{L+1}) + \alpha IP^m(y_{st}^e - D^{L+1})]. \quad (3)$$

The supplier's optimal echelon base-stock level  $y_s^m$  is the smallest minimizer of  $\mathcal{L}_s(\cdot)$ .

Under this echelon base-stock policy, firm  $j$  orders a sufficient amount in each period to raise its echelon inventory position  $x_{jt}^e$  to the echelon base-stock level  $y_j^m$  for  $j \in \{s, r\}$ . An equivalent optimal policy, known as an *installation base-stock policy*, is given by installation base-stock levels  $Y_s = \max(y_s^m - y_r^m, 0)$  and  $Y_r = \min(y_s^m, y_r^m)$ . Let  $x_{jt}$  and  $y_{jt}$  be firm  $j$ 's installation inventory position<sup>7</sup> before and after ordering in period  $t$ , respectively. Firm  $j$  orders a sufficient amount in each period to raise its installation inventory position  $x_{jt}$  to the installation base-stock level  $Y_j$  for  $j \in \{s, r\}$  (see, for example, Axsäter and Rosling 1993, Chen and Zheng 1994 for a discussion on the equivalence of installation and echelon base-stock policies for a serial system).

### 5. Local Control with Promised Lead-Time Contract

Under local control, the supplier offers a promised lead-time contract to the retailer. If the retailer accepts, the firms establish an inventory risk-sharing agreement for  $T$  periods. Next, the firms choose their respective optimal stocking levels. The remainder of the sequence of events is as described in §3.1. We use backward induction to characterize the optimal decisions; i.e., we solve for the optimal

inventory-stocking decisions, followed by the contracting decision.

#### 5.1. The Supplier's and the Retailer's Inventory Problem

For a given promised lead-time contract  $(\tau, K)$ , each firm minimizes its expected inventory cost over the next  $T$  periods. The per-period lump-sum payment  $K$  is independent of the order quantity and inventory on hand; hence, it has no effect on the inventory replenishment policy. Note also that the retailer is guaranteed on-time shipment of all orders under a promised lead-time contract. Hence, each firm independently solves a stationary, periodic-review inventory control problem with no upstream supply restrictions. The following dynamic programming recursion minimizes the cost of managing each firm's inventory over a finite horizon with  $T - t$  periods remaining until termination:

$$J_{jt}(x_{jt} | \tau) = \min_{y_{jt} \geq x_{jt}} \{G_j(y_{jt} | \tau) + \alpha E J_{j,t+1}(y_{jt} - D_t | \tau)\}$$

for all  $t \in \{1, \dots, T\}$ ,  $J_{j,T+1}(\cdot | \tau) \equiv 0^8$  for  $j \in \{s, r\}$ , where

$$\begin{aligned} G_s(y_{st} | \tau) &= (1 - \alpha)c_s y_{st} + E[h_s(y_{st} - D^{L+1-\tau})^+ \\ &\quad + p_s(D^{L+1-\tau} - y_{st})^+] \quad \text{and} \\ G_r(y_{rt} | \tau) &= (1 - \alpha)c_r y_{rt} + E[h_r(y_{rt} - D^{l+1+\tau})^+ \\ &\quad + p_r(D^{l+1+\tau} - y_{rt})^+]. \end{aligned}$$

For these stationary problems, a myopic base-stock policy is optimal (Veinott 1965). The optimal base-stock levels for the supplier and retailer are the minimizers of  $G_s(\cdot)$  and  $G_r(\cdot)$ , respectively:

$$\begin{aligned} Y_s^*(\tau) &= F_{L+1-\tau}^{-1} \left( \frac{p_s - (1 - \alpha)c_s}{h_s + p_s} \right) \quad \text{and} \\ Y_r^*(p_r, \tau) &= F_{l+1+\tau}^{-1} \left( \frac{p_r - (1 - \alpha)c_r}{h_r + p_r} \right). \end{aligned}$$

To have a reasonable solution, we assume that  $p_j \geq (1 - \alpha)c_j$  for firm  $j$ . Hence, with promised lead time  $\tau$ , firm  $j$  orders up to an optimal base-stock level  $Y_j^*(\tau)$  if its installation inventory position  $x_{jt}$  is below this level at the beginning of period  $t$ . The expected discounted inventory cost over  $T - t$  periods equals the sum of the discounted single-period costs, i.e.,

$$J_s(x_{st} | Y_s^*(\tau), \tau) = \sum_{k=t}^T \alpha^{k-t} G_s(Y_s^*(\tau) | \tau) = \theta G_s^*(\tau),$$

where

$$\begin{aligned} G_s^*(\tau) &\equiv c_s \mu + E\{h_s[Y_s^*(\tau) - D^{L+1-\tau}]^+ \\ &\quad + p_s[D^{L+1-\tau} - Y_s^*(\tau)]^+\}, \end{aligned}$$

$$J_r(x_{rt} | Y_r^*(p_r, \tau), \tau) = \sum_{k=t}^T \alpha^{k-t} G_r(Y_r^*(p_r, \tau) | \tau) \\ = \theta G_r^*(p_r, \tau),$$

where  $G_r^*(p_r, \tau) \equiv c_r \mu + E\{h_r[Y_r^*(p_r, \tau) - D^{t+1+\tau}]^+ + p_r[D^{t+1+\tau} - Y_r^*(p_r, \tau)]^+\}$  and  $\theta \equiv (1 - \alpha^{T-t+1})/(1 - \alpha)$ . The promised lead time  $\tau$  yields total expected inventory cost  $\theta[G_s^*(\tau) + G_r^*(p_r, \tau)]$  for the two-stage supply chain.

We simplify the expressions for the total expected inventory cost as follows. If we let  $p_r(c_r)$  be the unit penalty cost yielding the desired service level with unit variable cost  $c_r$ , then  $SL_r = (p_r(c_r) - (1 - \alpha)c_r)/(h_r + p_r(c_r))$ . By redefining  $p_r = h_r[p_r(c_r) - (1 - \alpha)c_r]/[h_r + (1 - \alpha)c_r]$ , we rewrite  $SL_r = p_r/(h_r + p_r)$ , so  $Y_r^*(p_r, \tau) = F_{t+1+\tau}^{-1}(p_r/(h_r + p_r))$ . Similarly, we rewrite  $Y_s^*(\tau) = F_{L+1-\tau}^{-1}(p_s/(h_s + p_s))$ . The terms  $c_s \mu$ ,  $c_r \mu$ , and  $\theta$  do not affect the optimization problems defined later in any structural way. Hence, we drop them from further consideration in what follows. Note that for a given holding cost, the service level implies a penalty cost and vice versa. Hence, we use these two terms interchangeably throughout the paper.

Next, we explore the properties of  $G_s^*(\tau)$  and  $G_r^*(p_r, \tau)$  that are necessary in determining the optimal contract terms later. To prove these properties, summarized in the next proposition, we consider a log-concave demand distribution  $F(\cdot)$  and use two stochastic ordering relationships for  $F_n(\cdot)$ , the  $n$ -fold convolution of demand distribution  $F(\cdot)$ . First,  $F_n(\cdot)$  is less than or equal to  $F_{n+1}(\cdot)$  in regular stochastic order, which implies  $F_n^{-1}(\beta) \leq F_{n+1}^{-1}(\beta)$  for all  $\beta \in (0, 1)$ . Second,  $F_n(\cdot)$  is less than or equal to  $F_{n+1}(\cdot)$  in dispersive order because the convolution of a log-concave density is log-concave (Theorem 2.B.3 of Shaked and Shanthikumar 1994). That is,  $F_n^{-1}(\gamma) - F_n^{-1}(\beta) \leq F_{n+1}^{-1}(\gamma) - F_{n+1}^{-1}(\beta)$  whenever  $0 < \beta \leq \gamma < 1$ . We define  $\Delta_\tau G_r^*(p_r, \tau) \equiv G_r^*(p_r, \tau) - G_r^*(p_r, \tau - 1)$ .

**PROPOSITION 1.** For a given promised lead time  $\tau$  and when  $F(\cdot)$  is log-concave: (a)  $\Delta_\tau G_r^*(p_r, \tau) \geq 0$ , (b)  $\partial G_r^*(p_r, \tau)/\partial p_r > 0$ , and (c)  $\partial \Delta_\tau G_r^*(p_r, \tau)/\partial p_r > 0$ .

Note that Normal and Erlang distributions, commonly used in inventory control, exhibit log-concavity (Bagnoli and Bergstrom 1989). From parts (a) and (b), the retailer's minimum expected inventory cost increases with both promised lead time  $\tau$  and backorder penalty cost  $p_r$ . Given the similarity between the minimum expected inventory cost functions for the supplier and the retailer, part (a) also implies that  $\Delta_\tau G_s^*(\tau) \leq 0$ . The supplier's minimum expected inventory cost decreases with  $\tau$ . From part (c), we observe an important property known as the *single-crossing property* (Fudenberg and Tirole 1991). Essentially, this property implies the following. A retailer that provides a high service level to end customers is more sensitive to an increase in promised lead time or, equivalently, benefits more from a decrease in promised lead time. This difference in sensitivity for an increase in promised lead time

enables a supplier to screen the retailer's private service-level information. This proposition is mainly used in the analysis of the asymmetric information case.

## 5.2. Full Information

When the supplier has full information about the retailer's service level, the optimal contract to offer to the retailer is the solution to the following problem:

$$\min_{\tau, K} G_s^*(\tau) - K \\ \text{s.t. } G_r^*(p_r, \tau) + K \leq U_r^{\max}, \\ \tau \in \{0, \dots, L + 1\}.$$

The supplier chooses a promised lead-time contract to minimize her expected inventory cost, while ensuring that the retailer's expected cost does not exceed his reservation cost  $U_r^{\max}$ . This cost could be the retailer's expected inventory cost under an existing contract or his outside option.

**PROPOSITION 2.** Under full information, the supplier optimally offers  $(\tau^f, K^f)$ , where

- (a)  $K^f = U_r^{\max} - G_r^*(p_r, \tau^f)$  and
- (b)  $\tau^f$  minimizes  $G_s^*(\tau) + G_r^*(p_r, \tau)$ .
- (c)  $\tau^f$  decreases<sup>9</sup> as  $p_r$  increases.
- (d)  $G_s^*(\tau^f) - K^f$  increases as  $p_r$  increases.

Part (a) states that the supplier minimizes her expected inventory cost by requiring the highest payment  $K^f$  that the retailer will accept. Part (b) states that the supplier optimally offers the first-best promised lead time. That is,  $\tau^f$  is the same  $\tau$  that would be chosen by a third party who optimizes the supply chain's inventory risk-sharing strategy. Note also from parts (c) and (d) that the supplier optimally offers a shorter promised lead time for a high-service retailer, and by doing so increases her own expected inventory cost.

## 5.3. Asymmetric Cost Information

The retailer's unit penalty cost  $p_r$  is often private information. Suppose that  $p_r$  lies within a finite set  $\{p_1, \dots, p_N\}$  of possible values, where  $p_1 < \dots < p_N$ . The retailer's unit penalty cost, his *type*, takes value  $p_i$  with probability  $\lambda_i$  for  $\sum_{i=1}^N \lambda_i = 1$ . This discrete distribution is public information.

**Learning the Service Level.** We investigate whether the supplier can learn the retailer's service level, either by asking the retailer or by direct observation of the retailer's past orders. If so, the supplier can solve for the full-information contract. Suppose that the supplier asks the retailer to report his service level. Suppose that the retailer reports his penalty cost as  $\hat{p}_r$  and the supplier offers the corresponding optimal full-information promised lead-time contract  $(\hat{\tau}^f, \hat{K}^f)$ . The retailer's actual expected cost under this contract is  $\hat{K}^f + G_r^*(p_r, \hat{\tau}^f)$ .

PROPOSITION 3. If  $\hat{p}_r > p_r$ , then  $\hat{\tau}^f \leq \tau^f$  and  $G_r^*(p_r, \hat{\tau}^f) + \hat{K}^f < G_r^*(p_r, \tau^f) + K^f$ .

The proposition states that a retailer who exaggerates his service level receives a shorter promised lead time, thereby shifting more inventory risk to the supplier. Plus, the retailer’s expected cost decreases, so he obtains a larger share of supply chain benefit under the contract. Hence, the retailer has reason to exaggerate his service-level information. The supplier, therefore, has reason not to trust the service-level information provided by the retailer.

A natural question to ask is whether the supplier can learn the retailer’s optimal service level over time by observing the retailer’s orders. This is not possible because the retailer optimally adopts a base-stock policy, and his order in each period is equal to the demand in the previous period. Hence, the supplier observes the demand, but she does not know how much the retailer satisfies from on-hand inventory and how much he backlogs. Hence, the supplier cannot learn the retailer’s service level over time. Next, we propose a mechanism that minimizes the supplier’s inventory-related cost, while enabling *credible* information sharing.

**Optimal Menu of Contracts.** Without observing the retailer’s service level, the supplier can minimize her inventory cost by designing a menu of promised lead-time contracts. According to the *revelation principle* (Myerson 1979), the supplier can limit her search for an optimal menu of contracts to the class of truth-telling contracts under which the retailer finds it optimal to reveal his unit penalty cost. To determine the optimal menu  $\{(\tau_i^a, K_i^a)_{i=1}^N\}$ , the supplier solves the following problem:

$$\begin{aligned} \min_{(\tau_i, K_i)_{i=1}^N} & \sum_{i=1}^N \lambda_i [G_s^*(\tau_i) - K_i] \\ \text{s.t. IR}_i: & G_r^*(p_i, \tau_i) + K_i \leq U_r^{\max} \quad \forall i \in \{1, \dots, N\}, \quad (5) \\ \text{IC}_{ij}: & G_r^*(p_i, \tau_i) + K_i \leq G_r^*(p_i, \tau_j) + K_j \quad \forall j \neq i, \\ & \tau_i \in \{0, \dots, L + 1\} \quad \forall i \in \{1, \dots, N\}. \end{aligned}$$

The individual rationality (IR) constraints in (5) guarantee that the retailer will find an acceptable contract in the menu. The incentive compatibility (IC) constraints ensure that a retailer with service level  $i$  voluntarily chooses the promised lead-time contract  $(\tau_i, K_i)$  designed for his true service-level type. We assume that when indifferent between  $(\tau_i, K_i)$  and  $(\tau_j, K_j)$  for  $j \neq i$ , a retailer with service level  $i$  chooses the former. Next, we use Proposition 1 to characterize an equivalent formulation for the feasible region defined by IR and IC. To simplify the exposition, we define  $\bar{\lambda}_j \equiv \sum_{i=1}^j \lambda_i$ ,  $\bar{\lambda}_0 \equiv 0$ , and  $p_0 \equiv p_1$ .

PROPOSITION 4. (a) A menu of contracts is feasible if and only if it satisfies the following conditions for all  $i \in \{1, \dots, N - 1\}$ : (i)  $\tau_i \geq \tau_{i+1}$ , and (ii)  $G_r^*(p_i, \tau_i) + K_i = U_r^{\max} - \sum_{k=i+1}^N [G_r^*(p_k, \tau_k) - G_r^*(p_{k-1}, \tau_k)]$  and  $G_r^*(p_N, \tau_N) + K_N = U_r^{\max}$ .

(b) The optimization problem in (5) has the following equivalent formulation:

$$\begin{aligned} \min_{\{\tau_1, \dots, \tau_N\}} & \sum_{i=1}^N \lambda_i \{ [G_s^*(\tau_i) + G_r^*(p_i, \tau_i)] + (\bar{\lambda}_{i-1} / \lambda_i) \\ & \cdot [G_r^*(p_i, \tau_i) - G_r^*(p_{i-1}, \tau_i)] \} - U_r^{\max} \quad (6) \\ \text{s.t. } & \tau_i \text{ is decreasing in } i, \\ & \tau_i \in \{0, \dots, L + 1\} \quad \forall i = 1, \dots, N. \end{aligned}$$

Part (a-i) states that the supplier optimally offers a shorter promised lead time to a retailer with a higher service level. Nevertheless, from part (a-ii), a retailer with the highest service level incurs his reservation cost. A retailer with a service level lower than  $p_N$  receives information rent, a cost reduction to discourage him from exaggerating his service level. These results are similar in nature to those obtained for other mechanism design problems (see, for example, Lovejoy 2006).

In part (b), the summation in the objective function (6) consists of two terms. The first term is the total supply chain inventory cost under the promised lead time. The second term  $(\bar{\lambda}_{i-1} / \lambda_i) [G_r^*(p_i, \tau_i) - G_r^*(p_{i-1}, \tau_i)]$  is the cost of the incentive problem (information rent) because of asymmetric service information. Without this term, the supplier would offer the first-best contract  $(\tau_i^f, K_i^f)$  to all retailer types.

Let  $\tau_i^a$  denote an optimal solution for the problem in (6), and let  $K_i^a$  denote the optimal payment that is obtained as the solution to the equalities in Proposition 4(a-ii) when  $\tau_i = \tau_i^a$ .

PROPOSITION 5. (a)  $K_i^a$  is increasing and  $\tau_i^a$  is decreasing in  $i$ .

- (b)  $G_r^*(p_i, \tau_i^a) + K_i^a$  is increasing in  $i$ .
- (c)  $\tau_1^a = \tau_1^f$  and  $\tau_i^a \leq \tau_i^f$  for all  $i \in \{2, \dots, N\}$ .

This proposition characterizes additional properties of the optimal menu of contracts. In particular, from part (a), a retailer with a higher service level optimally accepts a larger payment obligation for a shorter promised lead time. Because the optimal contract terms  $\tau_i^a$  and  $K_i^a$  are monotone in  $i$ , we can construct a function  $K(\tau)$  by setting  $K(\tau) = K_i^a$  if  $\tau = \tau_i^a$ . This function specifies a payment for a given promised lead time. The supplier offers this function as a contract without any mention of service levels. The retailer does not need to announce a service level either. This result makes the menu of promised lead-time contracts amenable for implementation. Part (c) states that all but the lowest service-type retailer gets a promised lead time shorter than the first-best promised lead time. Together with part (a), this implies two facts. First, the supplier faces more inventory risk when the retailer has private service-level information. Second, the supplier manages more of this inventory risk when doing business with a higher-service retailer.

#### 5.4. Cutoff Policy

The supplier may decide not to induce every retailer type to sign a promised lead-time contract when she wants to keep her expected inventory cost below  $U_s^{\max}$ . Next, we characterize an optimal cutoff-level policy (Corbett and de Groote 2000, Ha 2001). Under this policy, the supplier optimally induces the retailer to sign a promised lead-time contract if the retailer's backorder penalty cost is lower than a cutoff level  $\bar{p}$ . If the retailer's penalty cost (or equivalently, his service level) is higher, the two firms do not sign a promised lead-time contract, and hence do business under the original setting. To derive the supplier's optimal cutoff level, we introduce a null contract with promised lead time  $\tau \equiv \emptyset$ . Agreeing to the null contract is equivalent to no contract being signed, and expected inventory costs for the supplier and retailer revert to  $G_s^*(\emptyset) \equiv U_s^{\max}$  and  $G_r^*(\cdot, \emptyset) \equiv U_r^{\max}$ , respectively.

Under full information, the supplier determines the optimal cutoff level by solving (4) with an IR constraint for the supplier and a larger feasible set for the promised lead time, i.e.,

$$G_s^*(\tau) - K \leq U_s^{\max}, \quad \tau \in \{\emptyset, 0, \dots, L+1\}. \quad (7)$$

**PROPOSITION 6.** *Under full information, the supplier optimally offers the first-best contract  $(\tau, K) = (\tau^f, K^f)$  if  $p_r \leq \bar{p}$  and  $(\tau, K) = (\emptyset, 0)$  otherwise, where  $\bar{p} \equiv \min\{p \geq 0: G_s^*(\tau^f(p)) + G_r^*(p, \tau^f(p)) = U_s^{\max} + U_r^{\max}\}$ .*

The proposition shows that it is optimal for the supplier not to offer a promised lead-time contract to a retailer with  $p_r > \bar{p}$ . The proposition also provides an explicit solution for the cutoff level. Note that the cutoff level  $\bar{p}$  is increasing in both  $U_s^{\max}$  and  $U_r^{\max}$ . Intuitively, the supply chain has to carry more inventory to provide high service to the end customer. Hence, the supplier sets up a supply chain with a higher-service retailer only when either firm can accept a larger expected inventory cost.

Under asymmetric service-level information, the supplier's optimal menu of contracts, allowing for the null contract possibility, is the solution of the problem in (5) with

$$\text{IR}_s: \sum_{i=1}^N \lambda_i [G_s^*(\tau_i) - K_i] \leq U_s^{\max},$$

$$\tau_i \in \{\emptyset, 0, \dots, L+1\} \quad \forall i. \quad (8)$$

For notational convenience, we define sums over empty sets to be zero and  $\emptyset < 0$ .

**PROPOSITION 7.** *An optimal menu of contracts  $\{(\tau_i^c, K_i^c)_{i=1}^N\}$  satisfies the following properties: If  $\tau_i^c = \emptyset$ , then  $\tau_j^c = \emptyset$  and  $K_j^c = 0$  for  $j \geq i$ . In addition, if  $\tau_i^c \neq \emptyset$ , then  $\tau_j \geq \tau_i$  for  $j \leq i$ . Hence, an optimal cutoff level is defined as  $\bar{p} \equiv p_c$ ,*

where  $c = \max\{i \in 1, \dots, N \mid \tau_i^c \neq \emptyset\}$ . Let  $c = 0$  when this set is empty. Also,  $\bar{p}$  must satisfy

$$\left\{ \sum_{i=1}^c \lambda_i [U_s^{\max} + U_r^{\max} - G_s^*(\tau_i^c) - G_r^*(p_i, \tau_i^c)] \right\}$$

$$\geq \left\{ \sum_{i=1}^c \bar{\lambda}_{i-1} [G_r^*(p_i, \tau_i^c) - G_r^*(p_{i-1}, \tau_i^c)] \right\}.$$

Proposition 7 establishes the optimality of a cutoff policy. In other words, if it is optimal for the supplier not to offer a promised lead-time contract to a retailer with service level  $i$ , then it is also optimal not to offer a contract to a retailer with a higher service level. The optimal menu of contracts consists of  $(\tau_i, K_i) = (\emptyset, 0)$  for  $i > c$ . Hence, for  $p_i > \bar{p}$ , the two firms do not sign a promised lead-time contract, and the supplier and the retailer incur  $U_s^{\max}$  and  $U_r^{\max}$ , respectively.

#### 6. Comparing Stocking Levels

To understand the supplier's and retailer's deviation from a system-optimal solution, we compare the optimal base-stock levels under central control to those under local control with a promised lead time. The following proposition characterizes the optimal allocation of inventory under central control.

**PROPOSITION 8.** *Equations (1) and (3) have finite minimizers when  $p_r \geq P \equiv [(1 - \alpha)(c_s + \alpha^{L+1}c_r) + \alpha^L(1 - \alpha^{L+1})h_s]/\alpha^{L+1}$ . Then, the echelon and installation base-stock levels are given as follows:*

(a) For any  $p_r \in [P, \infty)$ ,  $y_r^m = F_{l+1}^{-1}((\alpha^l(p_r + h_s) - (1 - \alpha)c_r)/\alpha^l(p_r + h_r))$ .

(b) When  $p_r \in [P, H]$ , then  $y_s^m = F_{L+1+2}^{-1}((\alpha^{L+1}[\alpha^l(p_r + h_s) - (1 - \alpha)c_r] - (1 - \alpha)c_s - \alpha^L h_s)/\alpha^{L+1}(p_r + h_r))$ , where

$$H \equiv [(1 - \alpha)(c_s + \alpha^{L+1}c_r) + \alpha^L(1 - \alpha^{L+1})h_s$$

$$+ \alpha^{L+1}h_r F_{L+1+2}(y_r^m)]/[\alpha^{L+1}(1 - F_{L+1+2}(y_r^m))].$$

For this case,  $y_s^m \leq y_r^m$ . Hence,  $Y_s = 0$  and  $Y_r = y_s^m$ .

When  $p_r \in (H, \infty)$ , then  $y_s^m$  is the unique solution to

$$(1 - \alpha)c_s + \alpha^L h_s + \alpha^{L+1}[(1 - \alpha)c_r - \alpha^l(p_r + h_s)]$$

$$\cdot [1 - F_{L+1}(y - y_r^m)] + \alpha^{L+1}(p_r + h_r)$$

$$\cdot \int_{y-y_r^m}^{\infty} F_{l+1}(y-u)f_{L+1}(u) du = 0.$$

For this case,  $y_s^m > y_r^m$ . Hence,  $Y_s = y_s^m - y_r^m$  and  $Y_r = y_r^m$ .

Proposition 8 characterizes the optimal inventory-stocking levels at each location for a centrally controlled system and provides closed-form solutions under certain conditions. In particular, the optimal inventory allocation depends on how the penalty cost relates to the other cost parameters. When the penalty cost is very low, i.e., lower than  $P$ , the problem does not have a finite solution. When  $p_r \in [P, H]$ , the penalty cost is high enough to warrant holding inventory only at the retailer. When the penalty cost exceeds the threshold  $H$ , the penalty cost is high enough to warrant holding inventory at both locations.

PROPOSITION 9. For any given promised lead time  $\tau$ , the following are true:

(a) For  $p_r \in (H, \infty)$ , let

$$B(\tau) \equiv (p_r + h_r)F_{L+1} \left( F_{L+1+\tau}^{-1} \left( \frac{p_r - (1 - \alpha)c_r}{p_r + h_r} \right) \right) + \left( \frac{1 - \alpha}{\alpha^l} \right) c_r - p_r,$$

which is increasing in  $\tau$ . We have

$$\begin{cases} Y_r \leq Y_r^*(p_r, \tau) & \text{when } h_s \leq B(\tau), \\ Y_r > Y_r^*(p_r, \tau) & \text{when } h_s > B(\tau). \end{cases}$$

(b) For  $p_r \in [P, H]$ , we have  $Y_s^*(\tau) > Y_s = 0$  for  $\tau < L + 1$  and  $Y_s^*(\tau) = Y_s = 0$  for  $\tau = L + 1$ .

Proposition 9 states that if  $h_s$  is lower than  $B(\tau)$ , a central decision maker will hold less inventory at the retailer location than a locally controlled retailer would carry. In other words, under central control, when the holding cost at the supplier location is small, the central manager carries more inventory at the supplier location and less at the retailer location. However, under local control, the retailer does not consider the supplier’s inventory cost when making his inventory decision. Hence, there is a threshold for  $h_s$  below which a locally managed retailer carries more inventory than a centrally controlled one. Note also that this threshold increases with the promised lead time. Intuitively, a longer promised lead time requires a higher optimal inventory level for a locally controlled retailer because the retailer faces higher risk from holding insufficient inventory. Part (b) notes circumstances in which a local decision maker holds inventory at the supplier site, but a central decision maker does not. Similar comparisons are possible for other values of  $p_r$ .

These results show *when* and *how much* the supplier and the retailer may under- or overinvest in inventory when compared to a centrally controlled supply chain. Note that these comparisons are true for any given  $\tau$ . Hence, they apply to both full and asymmetric information scenarios. These solutions also enable one to quantify the difference in inventory levels due to decentralization through simple calculations. We provide more discussion on these comparisons in §8.

### 7. All-or-Nothing Inventory Allocation

Here, we address a special case in which the optimal promised lead time is either zero or  $L + 1$ . When the optimal promised lead time is zero, the supplier assumes all responsibility for the additional inventory risk. When the optimal promised lead time is  $L + 1$ , the supplier assumes none of this risk, essentially offering make-to-order service to the retailer. Hence, this outcome is referred to as an *all-or-nothing solution*.

PROPOSITION 10. Under full information and when  $G_s^*(\tau)$  and  $G_r^*(p_r, \tau)$  are concave in  $\tau$ , the supplier optimally offers  $(\tau^f, K^f)$ , where  $\tau^f \in \{0, L + 1\}$  and  $K^f = U_r^{\max} - G_r^*(p_r, \tau^f)$ .

For the supplier, concavity means that the incremental reduction in her expected cost increases as the promised lead time increases. For the retailer, concavity means that the incremental reduction in his expected cost increases as the promised lead time decreases. The optimal inventory risk-sharing strategy for the supply chain is then one of the extreme points of the feasible set of promised lead times, i.e., either zero or  $L + 1$ . The next proposition shows when the optimality of an all-or-nothing solution also holds under asymmetric information.

PROPOSITION 11. Under asymmetric information and when  $G_s^*(\tau)$ ,  $G_r^*(p_r, \tau)$ , and  $G_r^*(p_i, \tau_i) - G_r^*(p_{i-1}, \tau_i)$  are concave in  $\tau$ , the supplier’s optimal menu of contracts consists of all-or-nothing promised lead times, i.e.,  $\tau_i^a \in \{0, L + 1\}$  for all  $i$ . In particular, let  $m = \max\{i \in \{1, \dots, N\} \mid \tau_i^a = L + 1\}$ , and if this set is empty, let  $m = 0$ . Then, we have:

(a) If  $m > 0$ , then  $\tau_i^a = L + 1$  and  $K_i^a = [U_r^{\max} - G_r^*(p_m, L + 1)] - [G_r^*(p_N, 0) - G_r^*(p_m, 0)]$  for all  $i \leq m$ , whereas  $\tau_i^a = 0$  and  $K_i^a = U_r^{\max} - G_r^*(p_N, 0)$  for all  $i > m$ .

(b) Otherwise, if  $m = 0$ , then  $\tau_i^a = 0$  and  $K_i^a = U_r^{\max} - G_r^*(p_N, 0)$  for all  $i \in \{1, \dots, N\}$ .

Proposition 11 shows that a pooling equilibrium turns out to be an optimal outcome. In this equilibrium, the retailer does not necessarily reveal his backorder penalty cost by choosing a promised lead-time contract. In particular, part (a) states that if the retailer has a penalty cost lower than or equal to  $p_m$ , then he would optimally choose the contract with  $\tau^a = L + 1$ . Otherwise, he would select the contract with  $\tau^a = 0$ . Part (a) with  $m = N$  and part (b) together show the following. An optimal contract outcome could be a single contract that allocates all additional inventory risk to the supplier or to the retailer regardless of the retailer’s service level. Next, we investigate when the concavity conditions of Propositions 10 and 11 are met.

PROPOSITION 12.  $G_s^*(\tau)$ ,  $G_r^*(p_r, \tau)$ , and  $G_r^*(p_i, \tau_i) - G_r^*(p_{i-1}, \tau_i)$  are concave in  $\tau$  if: (i)  $F(\cdot)$  is Normal, or (ii)  $(f_n(F_n^{-1}(x)))^{-1}$  is concave in  $n$ . Any demand distribution satisfying one of these two conditions generates an all-or-nothing inventory risk-sharing agreement.

### 8. Numerical Examples

To illustrate the behavior of system cost and inventory levels under promised lead-time contracts, we present the following numerical examples. We set  $c_s = 0$ ,  $h_s = 1$  for the supplier and  $c_r = 0$ ,  $h_r = 3$  for the retailer. The retailer provides either a 85% (low) service level or a 98% (high) service level, stemming from unit penalty cost  $p_r = 17$  or  $p_r = 147$ , respectively. Demand is normally distributed with mean 50 and standard deviation 10, and the retailer’s reservation cost is  $U_r^{\max} = 150$ . We consider 27 scenarios that

differ in their production and processing lead times, cost of borrowing from an alternative source, and the likelihood of the retailer's providing a low versus a high service level. In each scenario, we make a combination of selections from the following sets:

$$(L, l) \in \{(2, 4), (3, 3), (4, 2)\}, \quad p_s \in \{19, 49, 99\},$$

$$\lambda \in \{0.2, 0.5, 0.8\}.$$

Throughout this section, an all-or-nothing promised lead time is optimal because demand is normally distributed (Proposition 12).

### 8.1. The Supplier's Cost Under the Promised Lead-Time Contract

In Figure 2, we plot the supplier's expected inventory cost as a function of lead times  $(L, l)$ , the supplier's shortage cost  $p_s$ , and  $\lambda$  under both full and asymmetric information. We highlight four observations.

First, the supplier's cost under full information is always lower than under asymmetric information. Compare, for example, the cost curve under full and asymmetric information when  $(L, l) = (4, 2)$ . Second, the supplier's expected cost increases with her cost of alternative sourcing, i.e., her penalty cost  $p_s$ . For example, under asymmetric information when  $\lambda = 0.8$  and  $(L, l) = (2, 4)$ , the supplier's expected cost is 66.37, 67.62, and 68.42 when  $p_s = 19, 49$ , and 99, respectively. Third, the supplier's expected cost is higher with lower  $\lambda$ . For example, under asymmetric information when  $p_s = 49$  and  $(L, l) = (2, 4)$ , the supplier's expected cost is 76.15, 71.89, and 67.62 when  $\lambda = 0.2, 0.5$ , and 0.8, respectively. This observation indicates that contracting with a low-service retailer is more profitable than contracting with a high-service retailer. Our fourth and

final observation is the decrease in the supplier's expected inventory cost when  $L$  increases for a constant  $L + l$ . A supplier located closer to the retailer incurs lower expected inventory cost under a promised lead-time contract. In other words, a supplier that is further away from the retailer faces more inventory risk. The following proposition proves this observation.

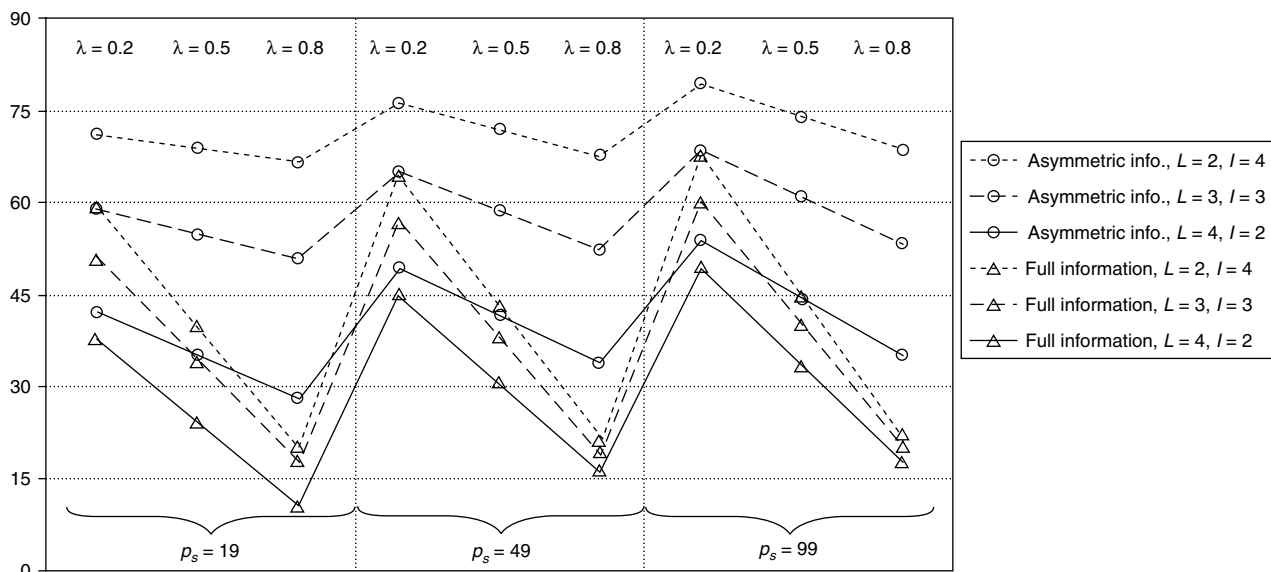
**PROPOSITION 13.** For constant system lead time  $L + l$ , normally distributed demand, and two possible retailer service levels, the supplier's expected inventory cost under asymmetric information increases as  $L$  decreases.

### 8.2. The Supplier's Stocking Level: Local vs. Central

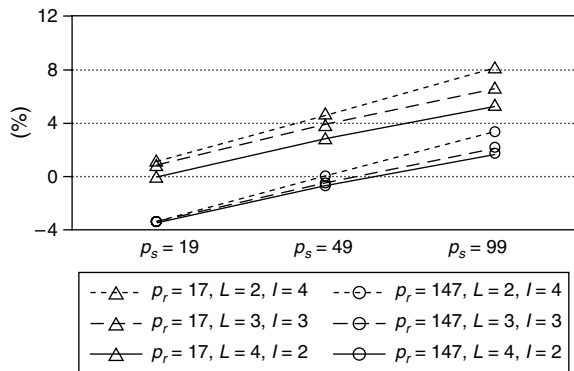
In the rest of this section, we compare the inventory levels under local control to those chosen by a central decision maker.

In Figure 3, we plot the percent increase in the supplier's inventory level under local control from the inventory level allocated under central control, i.e.,  $(Y_s^*(\tau) - Y_s)/Y_s * 100\%$ . This percent can be interpreted as over- (respectively, under-) investment in inventory due to local control when the percent is positive (respectively, negative). The promised lead time is zero for this figure. Consider, for example, the scenario in which  $p_r = 17, L = 2$ , and  $l = 4$ . When  $p_s = 49$ , the percentage increase is 4.7%. In other words, the supplier carries 4.7% more inventory as compared to a system-optimal inventory allocation. Note that the percent increase is also plotted as a function of the retailer's penalty cost, the shortage cost at the supplier, and the lead times. The following three observations are worth noting.

**Figure 2.** Supplier's expected inventory cost.



**Figure 3.** Percent increase in supplier inventory: Decentralized vs. centralized,  $\tau = 0$ .



First, the percent increase grows as the unit penalty cost  $p_s$  increases. Note that the points at which these lines cross zero, if they do, provide the supplier’s imputed penalty costs that equate the inventory allocation to the supplier’s site under local and central control. In other words, for those imputed shortage costs, the supplier chooses the system-optimal inventory level.

Second, as the penalty cost  $p_r$  increases or, equivalently, as the service level provided to end customers increases, the percent increase in supplier inventory due to local control decreases. In other words, the supplier’s overinvestment in inventory declines because under central control, more inventory is held at the supplier’s location (whereas under local control,  $p_r$  does not affect the supplier’s stocking level). The supplier may even underinvest in inventory when the retailer’s shortage cost is high. For example, when  $p_s = 49$ ,  $p_r = 147$ ,  $L = 4$ , and  $l = 2$ , the percent increase is  $-0.7\%$ . The supplier allocates  $0.7\%$  less inventory, hence, underinvests when compared to the system-optimal inventory allocation decision.

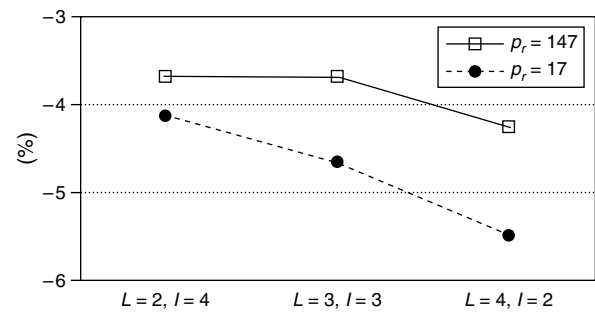
Finally, the percent increase in supplier inventory decreases as the supplier’s lead time  $L$  increases, whereas supply chain lead time  $L + l$  remains constant. In other words, the supplier’s overinvestment in inventory declines as the supplier is “located” closer to the end customer. This is mainly because the system-optimal solution starts to allocate more inventory to the supplier’s site as she gets closer to the customer.

**8.3. The Retailer’s Stocking Level: Local vs. Central**

Next, we illustrate how the retailer’s inventory level changes due to local control. In Figure 4, we plot the percent increase in the retailer’s inventory level under local control from the inventory level under central control, i.e.,  $(Y_r^*(\tau) - Y_r)/Y_r * 100\%$  for  $\tau = 0$ .

Note in Figure 4 that the retailer carries less inventory than a system-optimal inventory allocation. The retailer’s percent underinvestment in inventory decreases as either his service level  $p_r$ , or his processing lead time  $l$  increases

**Figure 4.** Percent increase in retailer inventory: Decentralized vs. centralized,  $\tau = 0$ .



(while keeping  $L + l$  constant). An increase in either  $p_r$  or  $l$  requires the retailer to carry more inventory to protect this second stage against larger demand uncertainty and more expensive shortage cost. Under central control, the manager has the option of holding some of this additional inventory at the supplier location. A retailer under local control does not have this option. Hence, the retailer increases his inventory level more dramatically than a central decision maker would have increased the inventory allocation to the second stage. As a result, the retailer’s inventory investment under local control approaches the level under central control as  $l$  or  $p_r$  increases.

**9. Conclusion**

We introduce a contract form that reduces a supplier’s uncertainty regarding demand and eliminates a retailer’s uncertainty regarding inventory availability: a promised lead-time contract. The contract and the model considered in this paper formalize what is already common in practice, that is, specifying lead times as part of a contract between two supply chain members (Billington 2002, Cohen et al. 2003).

In this study, by combining supply chain contracting, classical inventory control, and mechanism design, we discover the following. When the supplier has full information about the retailer’s inventory cost parameters, the optimal promised lead-time contract generates the optimal inventory risk-sharing strategy for the supply chain. Uncertainty regarding what service level the retailer provides to end customers generates conflict between the retailer and the supplier. In designing a mechanism to accommodate this uncertainty, the supplier offers shorter lead times and information rent. Altogether, this practice eats at the supplier’s contract benefits until, at some point, the supplier does not find it profitable to offer contracts to retailers with all possible service levels. In particular, this observation helps establish the optimality of a cutoff policy. Intuitively, retailers having a high service level may represent a market segment that is unprofitable for the supplier to serve under a promised lead-time contract. We also show that a centrally controlled supply chain holds more inventory at

the retailer's location than the retailer would carry independently when the holding cost at the supplier location is sufficiently high. Such comparisons show *when* and *how much* the supplier and the retailer under- or overinvest in inventory as compared to an integrated system managed by a central decision maker. The models can be used to quantify this difference in inventory investment and how this difference changes with the system parameters, such as lead times.

We also characterize conditions under which the supplier should assume all or none of the inventory risk. In particular, the optimal promised lead-time contract stipulates either build-to-stock ( $\tau = 0$ ) or build-to-order ( $\tau = L + 1$ ) service to the retailer under either full or asymmetric information when the single-period cost functions are concave in the promised lead time. A supplier working with multiple independent retailers can consider offering these two contracts and construct a portfolio of retailers with promised lead times zero and  $L + 1$ . We leave this topic to future research.

We assume that the supplier borrows inventory from an alternative source to satisfy retailer orders on time. Alternatively, if the supplier simply purchases emergency units from an outside source or expedites emergency units through overtime resources, the supplier pays a one-time unit emergency cost  $p_s$  above her normal production cost. This scenario results in the supplier facing the equivalent of a lost-sales inventory control problem: sales are lost for the supplier's normal production process. For a periodic-review problem under lost sales and positive lead times, Nahmias (1979) and references therein offer a two-term myopic heuristic, which can be used to extend the promised lead-time results to this alternate case.

The concept of risk sharing through promised lead-time contracts is a fertile avenue for future research. Some issues to explore include nonstationarities in cost or demand parameters, the impact of possible contract renegotiation, alternative information asymmetries, and the situation where the retailer takes the lead in contract development.

## 10. Electronic Companion

An electronic companion to this paper is available as part of the online version that can be found at <http://or.journal.informs.org/>.

## Appendix. Proofs

**PROOF OF PROPOSITION 1.** To prove part (a), recall that  $\mu$  is the mean demand for a single period. The retailer's minimum expected inventory cost can be written as

$$G_r^*(p_r, \tau) = p_r(l + 1 + \tau)\mu - (h_r + p_r) \int_0^{F_{l+1+\tau}^{-1}(p_r/(h_r+p_r))} u f_{l+1+\tau}(u) du. \quad (9)$$

Hence,

$$\begin{aligned} \Delta_\tau G_r^*(p_r, \tau) &= p_r\mu - (h_r + p_r) \left[ \int_0^{F_{l+1+\tau}^{-1}(p_r/(h_r+p_r))} u f_{l+1+\tau}(u) du \right. \\ &\quad \left. - \int_0^{F_{l+\tau}^{-1}(p_r/(h_r+p_r))} u f_{l+\tau}(u) du \right]. \end{aligned}$$

We divide this expression by  $(h_r + p_r)$  to obtain

$$\frac{\Delta_\tau G_r^*(p_r, \tau)}{h_r + p_r} = r\mu - \left[ \int_0^{F_{l+1+\tau}^{-1}(r)} u f_{l+1+\tau}(u) du - \int_0^{F_{l+\tau}^{-1}(r)} u f_{l+\tau}(u) du \right],$$

where  $r = p_r/(h_r + p_r)$ . To show that  $\Delta_\tau G_r^*(p_r, \tau) \geq 0$ , we show  $\Delta_\tau G_r^*(p_r, \tau)/(h_r + p_r) \geq 0$ . To do so, we first show that  $\Delta_\tau G_r^*(p_r, \tau)/(h_r + p_r)$  is concave in  $r$ . Note that  $(\partial^2/\partial r^2)(\Delta_\tau G_r^*(p_r, \tau)/(h_r + p_r)) = (f_{l+1+\tau}(F_{l+1+\tau}^{-1}(r)) - f_{l+\tau}(F_{l+\tau}^{-1}(r)))/f_{l+1+\tau}(F_{l+1+\tau}^{-1}(r))f_{l+\tau}(F_{l+\tau}^{-1}(r))$ .

Because  $F(\cdot)$  has a log-concave density,  $F_{l+\tau}$  is less than or equal to  $F_{l+1+\tau}$  in dispersive order. Hence, from Equation (2.B.7) in Shaked and Shanthikumar (1994), we have  $f_{l+1+\tau}(F_{l+1+\tau}^{-1}(r)) \leq f_{l+\tau}(F_{l+\tau}^{-1}(r))$  for all  $r$  in  $(0, 1)$ . Hence,  $(\partial^2/\partial r^2)(\Delta_\tau G_r^*(p_r, \tau)/(h_r + p_r)) \leq 0$ . Note that  $\Delta_\tau G_r^*(p_r, \tau)/(h_r + p_r)$  is continuous in  $r \in [0, 1]$  and equals zero for  $r \in \{0, 1\}$ . Together with concavity, this implies  $\Delta_\tau G_r^*(p_r, \tau)/(h_r + p_r) \geq 0$ .

To prove part (b), we note that

$$\begin{aligned} \frac{\partial}{\partial p_r} G_r^*(p_r, \tau) &= (l + 1 + \tau)\mu \\ &\quad - \int_0^{F_{l+1+\tau}^{-1}(r)} u f_{l+1+\tau}(u) du - (1 - r)F_{l+1+\tau}^{-1}(r). \end{aligned}$$

For  $r = 0$ ,  $(\partial/\partial p_r)G_r^*(p_r, \tau) = (l + 1 + \tau)\mu$ , and for  $r = 1$ ,  $\partial G_r^*(p_r, \tau)/\partial p_r = 0$ . Because

$$\partial^2 G_r^*(p_r, \tau)/\partial p_r \partial r = -(1 - r)/f_{l+1+\tau}(F_{l+1+\tau}^{-1}(r)) < 0,$$

we have  $\partial G_r^*(p_r, \tau)/\partial p_r > 0$ .

To prove (c), we note that

$$\begin{aligned} \frac{\partial \Delta_\tau G_r^*(p_r, \tau)}{\partial p_r} &= \mu - \left[ \int_0^{F_{l+1+\tau}^{-1}(r)} u f_{l+1+\tau}(u) du - \int_0^{F_{l+\tau}^{-1}(r)} u f_{l+\tau}(u) du \right] \\ &\quad - (1 - r)[F_{l+1+\tau}^{-1}(r) - F_{l+\tau}^{-1}(r)]. \end{aligned}$$

For  $r = 0$ ,  $\partial \Delta_\tau G_r^*(p_r, \tau)/\partial p_r = \mu$ , and for  $r = 1$ ,  $\partial \Delta_\tau G_r^*(p_r, \tau)/\partial p_r = 0$ . To prove  $\partial \Delta_\tau G_r^*(p_r, \tau)/\partial p_r > 0$ , we show that it is decreasing in  $r$ . To do so, note that

$$\begin{aligned} \frac{\partial^2 \Delta_\tau G_r^*(p_r, \tau)}{\partial p_r \partial r} &= -(1 - r) \left[ \frac{f_{l+\tau}(F_{l+\tau}^{-1}(r)) - f_{l+1+\tau}(F_{l+1+\tau}^{-1}(r))}{f_{l+\tau}(F_{l+\tau}^{-1}(r))f_{l+1+\tau}(F_{l+1+\tau}^{-1}(r))} \right]. \end{aligned}$$

The bracketed term on the right-hand side is positive because  $F_{l+\tau}$  is less than or equal to  $F_{l+1+\tau}$  in dispersive order.  $\square$

**PROOF OF PROPOSITION 2.** Note that  $G_r^*(p_r, \tau) + K = U_r^{\max}$  at optimality. Otherwise, one can increase  $K$  and reduce the objective function value without violating the constraint, contradicting the optimality of  $K$ . To prove part (b), substitute  $U_r^{\max} - G_r^*(p_r, \tau)$  for  $K$  in the objective function.

To prove parts (c) and (d), consider  $p_r < \hat{p}_r$  and let  $\tau^f$  and  $\hat{\tau}^f$  be the optimal promised lead times, respectively. Note from part (b) that  $G_s^*(\tau^f) + G_r^*(p_r, \tau^f) \leq G_s^*(\hat{\tau}^f) + G_r^*(p_r, \hat{\tau}^f)$  and  $G_s^*(\tau^f) + G_r^*(\hat{p}_r, \tau^f) \leq G_s^*(\hat{\tau}^f) + G_r^*(\hat{p}_r, \hat{\tau}^f)$ . By adding these two inequalities, we obtain  $G_r^*(\hat{p}_r, \tau^f) - G_r^*(p_r, \tau^f) \leq G_r^*(\hat{p}_r, \hat{\tau}^f) - G_r^*(p_r, \hat{\tau}^f)$ . By Proposition 1(c),  $\tau^f \geq \hat{\tau}^f$ . To prove part (d), from parts (a) and (b) and Proposition 1(b), we have  $G_s^*(\tau^f) - K^f = G_s^*(\tau^f) + G_r^*(p_r, \tau^f) - U_r^{\max} \leq G_s^*(\hat{\tau}^f) + G_r^*(p_r, \hat{\tau}^f) - U_r^{\max} < G_s^*(\hat{\tau}^f) + G_r^*(\hat{p}_r, \hat{\tau}^f) - U_r^{\max} = G_s^*(\hat{\tau}^f) - \hat{K}^f$ .  $\square$

**PROOF OF PROPOSITION 3.** The first statement follows from Proposition 2(c). To prove the second statement, we have  $G_r^*(p_r, \hat{\tau}^f) + \hat{K}^f < G_r^*(\hat{p}_r, \hat{\tau}^f) + \hat{K}^f = U_r^{\max} = G_r^*(p_r, \tau^f) + K^f$ . The inequality is from Proposition 1(b), and the equalities are from Proposition 2(a), concluding the proof.  $\square$

**PROOF OF PROPOSITION 4.** To prove part (a), we first show that IR and IC imply the conditions stated in the first part of the lemma. We begin by simplifying the IR constraint in (5). Suppose that  $i < j$ . We have  $G_r^*(p_i, \tau_i) + K_i \leq G_r^*(p_i, \tau_j) + K_j < G_r^*(p_j, \tau_j) + K_j \leq U_r^{\max}$ . The inequalities are due to IC, Proposition 1(b), and IR, respectively. Hence, IR is redundant for all  $i \neq N$ .

For  $i < j$ , we separate IC into upward constraints  $G_r^*(p_i, \tau_i) + K_i \leq G_r^*(p_i, \tau_j) + K_j$  and downward constraints  $G_r^*(p_j, \tau_j) + K_j \leq G_r^*(p_j, \tau_i) + K_i$ . By adding these two constraints, we have  $G_r^*(p_j, \tau_j) - G_r^*(p_i, \tau_j) \leq G_r^*(p_j, \tau_i) - G_r^*(p_i, \tau_i)$ . Hence, Proposition 1(c) implies  $\tau_j \leq \tau_i$ , showing part (a-i).

When  $i < j < k$ , we claim that  $IC_{ij}$  and  $IC_{jk}$  imply the upward  $IC_{ik}$  constraint. Assume for a contradiction that  $G_r^*(p_i, \tau_i) + K_i > G_r^*(p_i, \tau_k) + K_k$ , so  $K_k < K_i + G_r^*(p_i, \tau_i) - G_r^*(p_i, \tau_k)$ . The constraint  $IC_{jk}$  implies that  $K_j \leq K_k + G_r^*(p_j, \tau_k) - G_r^*(p_j, \tau_j)$ . Using the upward  $IC_{ij}$  constraint, we find that  $G_r^*(p_i, \tau_i) + K_i \leq G_r^*(p_i, \tau_j) + K_j \leq [K_k + G_r^*(p_j, \tau_k) - G_r^*(p_j, \tau_j)] + G_r^*(p_i, \tau_j) < [K_i + G_r^*(p_i, \tau_i) - G_r^*(p_i, \tau_k)] + G_r^*(p_j, \tau_k) - G_r^*(p_j, \tau_j) + G_r^*(p_i, \tau_j)$ . By rearranging terms, we have  $G_r^*(p_j, \tau_j) - G_r^*(p_i, \tau_j) < G_r^*(p_j, \tau_k) - G_r^*(p_i, \tau_k)$ , which implies  $\tau_j < \tau_k$  due to Proposition 1(c). However, this contradicts part (a-i), which we proved above. This result implies that the set of upward constraints reduces to adjacent upward constraints  $G_r^*(p_i, \tau_i) + K_i \leq G_r^*(p_i, \tau_{i+1}) + K_{i+1}$  for  $i \in \{1, \dots, N-1\}$ .

Next, we show that the remaining upward constraints are binding and the downward constraints are redundant at optimality. To do so, we use an induction argument.

First, we show that  $G_r^*(p_i, \tau_i) + K_i = G_r^*(p_i, \tau_{i+1}) + K_{i+1}$  for  $i = 1$ . Assume for a contradiction that  $G_r^*(p_1, \tau_1) + K_1 < G_r^*(p_1, \tau_2) + K_2$ . Proposition 1(b) and IR for  $i = 2$  imply that  $K_1 < K_2 + G_r^*(p_1, \tau_2) - G_r^*(p_1, \tau_1) < K_2 + G_r^*(p_2, \tau_2) - G_r^*(p_1, \tau_1) \leq U_r^{\max} - G_r^*(p_1, \tau_1)$ . Hence, one can increase  $K_1$  and lower the objective function value without violating other constraints, contradicting optimality. Hence, IC for  $i = 1$  must be binding.

Next, assume for an induction argument that  $G_r^*(p_i, \tau_i) + K_i = G_r^*(p_i, \tau_{i+1}) + K_{i+1}$  for all  $i \in \{1, \dots, j-1\}$ , where  $j \geq 2$ . We show that these binding upward constraints and  $\tau_{i+1} \leq \tau_i$  imply downward IC constraints  $G_r^*(p_j, \tau_j) + K_j \leq G_r^*(p_j, \tau_i) + K_i$  for all  $i < j$ . Note that  $K_{i+1} = K_i + [G_r^*(p_i, \tau_i) - G_r^*(p_i, \tau_{i+1})]$ , and for  $j > i$  iteratively solve for  $K_j = K_i + \sum_{k=i}^{j-1} [G_r^*(p_k, \tau_k) - G_r^*(p_k, \tau_{k+1})]$ . Proposition 1(c) implies  $G_r^*(p_j, \tau_j) - G_r^*(p_{j-1}, \tau_j) \leq G_r^*(p_j, \tau_{j-1}) - G_r^*(p_{j-1}, \tau_{j-1})$ , and adding this inequality to the identity for  $K_j$  yields  $G_r^*(p_j, \tau_j) + K_j \leq K_i + \sum_{k=i}^{j-2} [G_r^*(p_k, \tau_k) - G_r^*(p_k, \tau_{k+1})] + G_r^*(p_j, \tau_{j-1})$ . Proposition 1(c) also implies the sequence of inequalities  $\{G_r^*(p_{j-m-1}, \tau_{j-m-1}) - G_r^*(p_{j-m-1}, \tau_{j-m}) \leq G_r^*(p_j, \tau_{j-m-1}) - G_r^*(p_j, \tau_{j-m})\}_{m=1}^{j-i-1}$ . Iteratively adding these to  $K_j + G_r^*(p_j, \tau_j) \leq K_i + \sum_{k=i}^{j-2} [G_r^*(p_k, \tau_k) - G_r^*(p_k, \tau_{k+1})] + G_r^*(p_j, \tau_{j-1})$  yields  $G_r^*(p_j, \tau_j) + K_j \leq G_r^*(p_j, \tau_i) + K_i$ ; hence, downward constraints are redundant for all  $i < j$ .

To conclude the induction argument, we show that the upward adjacent constraint for  $j$  is also binding. Assume for a contradiction that it is not, i.e.,  $G_r^*(p_j, \tau_j) + K_j < G_r^*(p_j, \tau_{j+1}) + K_{j+1}$ . Proposition 1(b) and IR for  $j+1$  imply that  $K_j < K_{j+1} + G_r^*(p_j, \tau_{j+1}) - G_r^*(p_j, \tau_j) < K_{j+1} + G_r^*(p_{j+1}, \tau_{j+1}) - G_r^*(p_j, \tau_j) \leq U_r^{\max} - G_r^*(p_j, \tau_j)$ . Hence, one can increase  $K_j$  and lower the objective function value without violating any constraint. This contradicts optimality. Hence, all remaining upward IC are binding at optimality, and they can be replaced by

$$IC': G_r^*(p_i, \tau_i) + K_i = G_r^*(p_i, \tau_{i+1}) + K_{i+1}$$

$$\text{for } i \in \{1, \dots, N-1\}.$$

Note also that  $G_r^*(p_N, \tau_N) + K_N = U_r^{\max}$  at optimality. Otherwise, one can increase  $K_N$  by  $\epsilon > 0$  and all other  $K_i$  by the same amount, thereby reducing the objective function value without violating any constraints. This contradicts optimality. Hence, IR constraints can be replaced by

$$IR': G_r^*(p_N, \tau_N) + K_N = U_r^{\max}.$$

Note that  $IC'$  and  $IR'$  imply part (a-ii).

Conversely, we show that the two conditions (i) and (ii) imply IR and IC. Note that condition (ii) implies IR because  $G_r^*(p_k, \tau_k) > G_r^*(p_{k-1}, \tau_k)$  from Proposition 1(b). Next, we inductively show that (i) and (ii) imply upward IC constraints. Condition (ii) implies  $K_i = K_{i+1} + G_r^*(p_i, \tau_{i+1}) - G_r^*(p_i, \tau_i)$  for all  $i \in \{1, \dots, N-1\}$ .

Assume for an induction argument that  $G_r^*(p_i, \tau_i) + K_i \leq G_r^*(p_i, \tau_j) + K_j$  for some  $j > i$ . This inequality holds for  $i = 1$  and  $j = 2$ . We have

$$\begin{aligned} & G_r^*(p_i, \tau_j) + K_j \\ &= G_r^*(p_i, \tau_j) + [K_{j+1} + G_r^*(p_j, \tau_{j+1}) - G_r^*(p_j, \tau_j)] \\ &\quad + [G_r^*(p_i, \tau_{j+1}) - G_r^*(p_i, \tau_j)] \\ &= G_r^*(p_i, \tau_{j+1}) + K_{j+1} + \{[G_r^*(p_i, \tau_j) - G_r^*(p_i, \tau_{j+1})] \\ &\quad - [G_r^*(p_j, \tau_j) - G_r^*(p_j, \tau_{j+1})]\}. \end{aligned}$$

The term in brackets is negative from Proposition 1(c) and condition (i)  $\tau_j \geq \tau_{j+1}$ . Hence, we have  $G_r^*(p_i, \tau_i) + K_i \leq G_r^*(p_i, \tau_j) + K_j \leq G_r^*(p_i, \tau_{j+1}) + K_{j+1}$ , concluding the induction argument. Note that (ii) also implies IC'. Above we also showed that (i) and IC' imply downward IC constraints. This concludes the proof for part (a).

To prove part (b), we substitute for  $K_i$  and use part (a) to obtain (6).  $\square$

**PROOF OF PROPOSITION 5.** Part (a) follows from Proposition 4(a). In particular,  $\tau_i^a$  must be decreasing in  $i$  to be a feasible solution. Now assume for a contradiction that  $K_{i+1}^a < K_i^a$ . Together with Proposition 4(aii), this assumption implies  $K_{i+1}^a = U_r^{\max} - G_r^*(p_{i+1}, \tau_{i+1}^a) - \sum_{k=i+2}^N [G_r^*(p_k, \tau_k^a) - G_r^*(p_{k-1}, \tau_k^a)] < U_r^{\max} - G_r^*(p_i, \tau_i^a) - \sum_{k=i+1}^N [G_r^*(p_k, \tau_k^a) - G_r^*(p_{k-1}, \tau_k^a)] = K_i^a$ . However, by canceling terms, this inequality yields  $G_r^*(p_i, \tau_i^a) < G_r^*(p_i, \tau_{i+1}^a)$ , which contradicts Proposition 1(a). Hence,  $K_{i+1}^a \geq K_i^a$ .

To prove part (b), note that Proposition 4(a-ii) defines the retailer's resulting expected cost. From Proposition 1(b), each term in  $\sum_{k=i+1}^N [G_r^*(p_k, \tau_k) - G_r^*(p_{k-1}, \tau_k)]$  is positive, which implies part (b).

To prove part (c), note that  $\tau_{i+1}^a \leq \tau_i^a$ . Hence, exactly one of the following is true: (i)  $\tau_i^a \leq \tau_i^f$ , (ii)  $\tau_{i+1}^a \leq \tau_i^f < \tau_i^a$ , (iii)  $\tau_{j+1}^a \leq \tau_i^f < \tau_j^a$  for some  $j \in \{i+1, \dots, N-1\}$ , or (iv)  $\tau_i^f < \tau_N^a$ . To prove  $\tau_i^a \leq \tau_i^f$ , we will show that the other possibilities lead to a contradiction.

First, assume for a contradiction that  $\tau_{i+1}^a \leq \tau_i^f < \tau_i^a$ . Then,  $\{\tau_N^a, \dots, \tau_{i+1}^a, \tau_i^f, \tau_{i-1}^a, \dots, \tau_1^a\}$  is a feasible solution to the problem in (6). From optimality, we have

$$\begin{aligned} & \sum_{k \neq i, k=1}^N \{ \lambda_k [G_s^*(\tau_k^a) + G_r^*(p_k, \tau_k^a)] \\ & \quad + \bar{\lambda}_{k-1} [G_r^*(p_k, \tau_k^a) - G_r^*(p_{k-1}, \tau_k^a)] \} \\ & + \{ \lambda_i [G_s^*(\tau_i^a) + G_r^*(p_i, \tau_i^a)] \\ & \quad + \bar{\lambda}_{i-1} [G_r^*(p_i, \tau_i^a) - G_r^*(p_{i-1}, \tau_i^a)] \} \\ & \leq \sum_{k \neq i, k=1}^N \{ \lambda_k [G_s^*(\tau_k^a) + G_r^*(p_k, \tau_k^a)] \\ & \quad + \bar{\lambda}_{k-1} [G_r^*(p_k, \tau_k^a) - G_r^*(p_{k-1}, \tau_k^a)] \} \\ & + \{ \lambda_i [G_s^*(\tau_i^f) + G_r^*(p_i, \tau_i^f)] \\ & \quad + \bar{\lambda}_{i-1} [G_r^*(p_i, \tau_i^f) - G_r^*(p_{i-1}, \tau_i^f)] \}. \end{aligned} \quad (10)$$

Note also that optimality of  $\tau_i^f$  under full information implies  $\lambda_i [G_s^*(\tau_i^f) + G_r^*(p_i, \tau_i^f)] \leq \lambda_i [G_s^*(\tau_i^a) + G_r^*(p_i, \tau_i^a)]$ . Adding the last two inequalities yields  $[G_r^*(p_i, \tau_i^a) - G_r^*(p_{i-1}, \tau_i^a)] \leq [G_r^*(p_i, \tau_i^f) - G_r^*(p_{i-1}, \tau_i^f)]$ , which contradicts Proposition 1(c). Therefore, we cannot have  $\tau_{i+1}^a \leq \tau_i^f < \tau_i^a$ .

Second, assume that the statement  $\tau_{j+1}^a \leq \tau_i^f < \tau_j^a$  for some  $j \in \{i+1, \dots, N-1\}$  is true. Then,  $\{\tau_N^a, \dots, \tau_{j+1}^a, \tau_i^f, \tau_{j-1}^a, \dots, \tau_1^a\}$  is a feasible solution to the problem in (6). From optimality, we have

$$\begin{aligned} & \lambda_j [G_s^*(\tau_j^a) + G_r^*(p_j, \tau_j^a)] \\ & \quad + \bar{\lambda}_{j-1} [G_r^*(p_j, \tau_j^a) - G_r^*(p_{j-1}, \tau_j^a)] \\ & \leq \lambda_j [G_s^*(\tau_i^f) + G_r^*(p_j, \tau_i^f)] \\ & \quad + \bar{\lambda}_{j-1} [G_r^*(p_j, \tau_i^f) - G_r^*(p_{j-1}, \tau_i^f)]. \end{aligned}$$

Optimality of  $\tau_i^f$  under full information implies

$$\lambda_j [G_s^*(\tau_i^f) + G_r^*(p_i, \tau_i^f)] \leq \lambda_j [G_s^*(\tau_j^a) + G_r^*(p_i, \tau_j^a)].$$

Adding the last two inequalities and rearranging terms yields

$$\begin{aligned} & \lambda_j [G_r^*(p_j, \tau_j^a) - G_r^*(p_i, \tau_j^a)] \\ & \quad + \bar{\lambda}_{j-1} [G_r^*(p_j, \tau_j^a) - G_r^*(p_{j-1}, \tau_j^a)] \\ & \leq \lambda_j [G_r^*(p_j, \tau_i^f) - G_r^*(p_i, \tau_i^f)] \\ & \quad + \bar{\lambda}_{j-1} [G_r^*(p_j, \tau_i^f) - G_r^*(p_{j-1}, \tau_i^f)]. \end{aligned} \quad (11)$$

Because  $\tau_j^a > \tau_i^f$ ,  $p_j > p_{j-1}$ , and  $p_j > p_i$ , Proposition 1(c) implies the following two relationships:

$$\begin{aligned} & G_r^*(p_j, \tau_j^a) - G_r^*(p_{j-1}, \tau_j^a) > G_r^*(p_j, \tau_i^f) - G_r^*(p_{j-1}, \tau_i^f), \\ & G_r^*(p_j, \tau_j^a) - G_r^*(p_i, \tau_j^a) > G_r^*(p_j, \tau_i^f) - G_r^*(p_i, \tau_i^f). \end{aligned}$$

Therefore, the left side of (11) is greater than the right side, and we have a contradiction. Hence, we cannot have  $\tau_{j+1}^a \leq \tau_i^f < \tau_j^a$  for some  $j \in \{i+1, \dots, N-1\}$ .

Finally, we assume that the statement  $\tau_i^f < \tau_N^a$  is true. Then,  $\{\tau_i^f, \tau_{N-1}^a, \dots, \tau_1^a\}$  is a feasible solution to the problem in (6). Arguments similar to those in the previous case yield a contradiction with Proposition 1(c). Hence, it must be true that  $\tau_i^a \leq \tau_i^f$ .

To prove  $\tau_1^a = \tau_1^f$ , note that  $\{\tau_1^f, \tau_2^a, \dots, \tau_N^a\}$  is a feasible solution to the problem in (6) because  $\tau_1^a \leq \tau_1^f$ . Optimality of  $\{\tau_1^a, \dots, \tau_N^a\}$  yields  $\lambda_1 [G_s^*(\tau_1^a) + G_r^*(p_1, \tau_1^a)] \leq \lambda_1 [G_s^*(\tau_1^f) + G_r^*(p_1, \tau_1^f)]$  from Equation (10) with  $i = 1$ . Optimality of  $\tau_1^f$  under full information yields  $\lambda_1 [G_s^*(\tau_1^f) + G_r^*(p_1, \tau_1^f)] \leq \lambda_1 [G_s^*(\tau_1^a) + G_r^*(p_1, \tau_1^a)]$ . Hence,  $[G_s^*(\tau_1^a) + G_r^*(p_1, \tau_1^a)] = [G_s^*(\tau_1^f) + G_r^*(p_1, \tau_1^f)]$ , so  $\tau_1^a = \tau_1^f$  is an optimal solution.  $\square$

**PROOF OF PROPOSITION 6.** At optimality, we have  $K = U_r^{\max} - G_r^*(p_r, \tau(p_r))$ . Hence,  $K = 0$  when  $\tau = \emptyset$ . Then,

problem (4) with (7) is equivalent to

$$\begin{aligned} \min_{\tau} \quad & G_s^*(\tau(p_r)) + G_r^*(p_r, \tau(p_r)) - U_r^{\max} \\ \text{s.t.} \quad & G_s^*(\tau(p_r)) + G_r^*(p_r, \tau(p_r)) \leq U_r^{\max} + U_s^{\max}, \\ & \tau \in \{\emptyset, 0, \dots, L+1\}. \end{aligned}$$

Proposition 2(a) and 2(d) together imply that  $G_s^*(\tau^f(p_r)) + G_r^*(p_r, \tau^f(p_r))$  increases with  $p_r$ . Hence, there exists a threshold  $\bar{p}$  such that for  $p_r \leq \bar{p}$ , contract  $(\tau^f, K^f)$  is feasible and optimal, whereas for  $p_r > \bar{p}$ , the null contract  $(\emptyset, 0)$  is optimal.  $\square$

**PROOF OF PROPOSITION 7.** First, we show that if  $\tau_i^c = \emptyset$  for some  $i \in \{1, \dots, N-1\}$ , then  $\tau_j^c = \emptyset$  for all  $j > i$ . Assume for a contradiction that  $\tau_j^c \neq \emptyset$ . Adding upward constraints  $G_r^*(p_i, \tau_i^c) + K_i^c \leq G_r^*(p_j, \tau_j^c) + K_j^c$  and the downward constraints  $G_r^*(p_j, \tau_j^c) + K_j^c \leq G_r^*(p_j, \tau_i^c) + K_i^c$ , we have  $G_r^*(p_j, \tau_j^c) - G_r^*(p_i, \tau_i^c) \leq G_r^*(p_j, \tau_i^c) - G_r^*(p_i, \tau_i^c)$ . From Proposition 1(b),  $G_r^*(p_j, \tau_j^c) - G_r^*(p_i, \tau_i^c) > 0$ . Hence,  $G_r^*(p_j, \tau_i^c) - G_r^*(p_i, \tau_i^c) > 0$ . However, this contradicts  $\tau_i^c = \emptyset$  because  $G_r^*(\cdot, \tau_i^c) \equiv U_r^{\max}$ , which implies  $0 > 0$ . Hence, it must be true that  $\tau_j^c = \emptyset$ .

Note that the above result also implies that if  $\tau_j^c \neq \emptyset$  for some  $j \in \{1, \dots, N-1\}$ , then  $\tau_i^c \neq \emptyset$  for all  $i < j$ . Hence, for such  $i < j$  pairs, adding upward constraints and downward constraints and rearranging terms as above yields  $G_r^*(p_j, \tau_j^c) - G_r^*(p_i, \tau_i^c) \leq G_r^*(p_j, \tau_i^c) - G_r^*(p_i, \tau_i^c)$ . This inequality, together with Proposition 1(c), implies  $\tau_j^c \leq \tau_i^c$ .

Now let  $c = \max\{i \in \{1, \dots, N\} \mid \tau_i^c \neq \emptyset\}$  at optimality. Note that if this set is empty, the problem is trivial. In other words, if  $\tau_i^c = \emptyset$  for all  $i$ , then all IR and IC constraints reduce to  $K_i^c \leq 0$ . Hence, at optimality,  $K_i^c = 0$ . Otherwise, the supplier can increase the  $K_i^c$ 's to zero and reduce the objective function value.

If the set is not empty, the rest of the proof follows exactly the same arguments of the proof of Proposition 4 for  $i \leq c$ . The differences in the proofs are for the boundary cases when  $i > c$ . In what follows, we will only provide the details for those cases.

For example, IR constraints for  $j > c$ , e.g.,  $G_r^*(p_j, \emptyset) + K_j^c \leq U_r^{\max}$ , imply that  $K_j^c \leq 0$  for all  $j > c$ . Consider  $i > c$ . Upward IC constraints  $G_r^*(p_i, \emptyset) + K_i^c \leq G_r^*(p_i, \emptyset) + K_j^c$  imply  $K_i^c \leq K_j^c$  for any  $j > i$ . Downward IC constraints  $G_r^*(p_i, \emptyset) + K_i^c \leq G_r^*(p_i, \emptyset) + K_k^c$  imply  $K_i^c \leq K_k^c$  for any  $k < i$  and  $k > c$ . Hence, we can replace all upward and downward constraints and IR for  $i > c$  with  $K_i^c = K \leq 0$ .

The set of IR constraints for the retailer with  $i < c$  are redundant. In addition, IR for  $i = c$  is also redundant because  $G_r^*(p_c, \tau_c^c) + K_c^c \leq G_r^*(p_c, \tau_{c+1}^c) + K = U_r^{\max} + K \leq U_r^{\max}$ . The first and second inequalities are from IC and  $K_j^c = K \leq 0$  for  $j > c$ , respectively.

The upward IC constraints can be reduced to adjacent upward IC constraints for all  $i < c$ . For  $i \leq c < j$ , note that  $G_r^*(p_i, \tau_i^c) + K_i^c \leq G_r^*(p_i, \tau_j^c) + K = G_r^*(p_i, \tau_k^c) + K$  for all  $k > j$ . For  $k \in (i, c]$ , we also have  $G_r^*(p_i, \tau_i^c) + K_i^c \leq$

$G_r^*(p_i, \tau_k^c) + K_k^c < G_r^*(p_k, \tau_k^c) + K_k^c \leq G_r^*(p_k, \tau_j^c) + K = G_r^*(p_i, \tau_j^c) + K$ . The inequalities are due to IC<sub>ik</sub>, Proposition 1(b), and IC<sub>kj</sub>, respectively. The equality is because  $\tau_j^c = \emptyset$  for  $j > c$ . Hence, all upward IC constraints can be reduced to adjacent upward IC constraints.

The adjacent IC are binding and the downward IC constraints are redundant for all  $i, j \leq c$ . For  $i \leq c < c+1 < j$ , we show that IC<sub>c+1,i</sub> implies IC<sub>ji</sub>. We have  $G_r^*(p_j, \tau_j^c) + K = G_r^*(p_{c+1}, \tau_{c+1}^c) + K \leq G_r^*(p_{c+1}, \tau_i^c) + K_i^c < G_r^*(p_j, \tau_i^c) + K_i^c$ . The first equality is due to  $\tau_j^c = \tau_{c+1}^c = \emptyset$ . The inequalities are due to IC and Proposition 1(b), respectively. Next, we show that downward IC<sub>c+1,i</sub> for  $i < c$  is also redundant. Note that  $G_r^*(p_{c+1}, \tau_{c+1}^c) + K \leq G_r^*(p_{c+1}, \tau_c^c) + K_c^c \leq G_r^*(p_{c+1}, \tau_c^c) + K_i^c + [G_r^*(p_c, \tau_i^c) - G_r^*(p_c, \tau_c^c)] \leq G_r^*(p_{c+1}, \tau_c^c) + K_i^c + [G_r^*(p_{c+1}, \tau_i^c) - G_r^*(p_{c+1}, \tau_c^c)] = G_r^*(p_{c+1}, \tau_i^c) + K_i^c$ . The first and second inequalities are due to IC<sub>c+1,c</sub> and IC<sub>ci</sub>, respectively. The last inequality is from Proposition 1(c) and  $\tau_i^c \geq \tau_c^c$ .

Next, note that IC<sub>c,c+1</sub> must be binding. Assume for a contradiction that  $G_r^*(p_c, \tau_c^c) + K_c^c < G_r^*(p_c, \tau_{c+1}^c) + K$ . This implies  $K_c^c < K + U_r^{\max} - G_r^*(p_c, \tau_c^c)$ . Then, the supplier can increase  $K_c^c$  to  $K + U_r^{\max} - G_r^*(p_c, \tau_c^c)$  without violating any constraints, which contradicts optimality. Note also that IC<sub>c,c+1</sub> binding implies IC<sub>c+1,c</sub>. We have  $G_r^*(p_{c+1}, \tau_{c+1}^c) + K = G_r^*(p_c, \tau_{c+1}^c) + K = G_r^*(p_c, \tau_c^c) + K_c^c < G_r^*(p_{c+1}, \tau_c^c) + K_c^c$ . The first and second equalities are from  $\tau_{c+1}^c = \emptyset$  and the binding IC<sub>c,c+1</sub>, respectively. The inequality is due to Proposition 1(b).

From binding upward adjacent IC, we have  $K_i^c = K_{i+1}^c - [G_r^*(p_i, \tau_i^c) - G_r^*(p_i, \tau_{i+1}^c)]$  for all  $i < c$ , and  $K_c^c = K + U_r^{\max} - G_r^*(p_c, \tau_c^c)$ , and  $K_j^c = K \leq 0$  for all  $j > c$ . The expressions for  $K_c^c$  and  $K_i^c$  for all  $i < c$  together imply  $K_i^c = K + [U_r^{\max} - G_r^*(p_i, \tau_i^c)] - \sum_{k=i+1}^c [G_r^*(p_k, \tau_k^c) - G_r^*(p_{k-1}, \tau_k^c)]$  for all  $i < c$ .

Assume for a contradiction that  $K < 0$ . Then, the supplier can increase  $K$  to zero, and hence increase  $K_i^c$  for all  $i > c$  by the same amount, without violating any constraints. This contradicts optimality. Therefore,  $K_j^c = K = 0$  for all  $j > c$ . Hence,  $G_r^*(p_c, \tau_c^c) + K_c^c = U_r^{\max}$  and  $G_r^*(p_i, \tau_i^c) + K_i^c = U_r^{\max} - \sum_{k=i+1}^c [G_r^*(p_k, \tau_k^c) - G_r^*(p_{k-1}, \tau_k^c)]$  for all  $i < c$ . Iteratively solving for  $K_i^c$  and plugging it into the objective function yields the optimal solution  $\sum_{i=1}^c \lambda_i \{ [G_s^*(\tau_i^c) + G_r^*(p_i, \tau_i^c) - U_r^{\max}] + \bar{\lambda}_{i-1} / \lambda_i [G_r^*(p_i, \tau_i^c) - G_r^*(p_{i-1}, \tau_i^c)] \} + \sum_{i=c+1}^N \lambda_i U_s^{\max} \leq U_s^{\max}$ . The inequality is due to the IR<sub>s</sub>. This inequality implies the condition on the cutoff level presented in the statement of the proposition.

**PROOF OF PROPOSITION 8.** To prove part (a), from Equation (1), we have  $(\partial/\partial y)\mathcal{L}_r(y) = (1-\alpha)c_r - \alpha^l(p_r + h_s) + \alpha^l(p_r + h_r)F_{l+1}(y)$ , which is increasing in  $y$ . Because  $p_r \geq P$  implies  $\alpha^l(p_r + h_s) - (1-\alpha)c_r \geq 0$ , we have  $(\partial/\partial y)\mathcal{L}_r(0) = (1-\alpha)c_r - \alpha^l(p_r + h_s) \leq 0$ . Note also that  $\lim_{y \rightarrow \infty} (\partial/\partial y)\mathcal{L}_r(y) = (1-\alpha)c_r + \alpha^l(h_r - h_s) > 0$ . Hence, there exists a positive finite  $y$  such that  $(\partial/\partial y)\mathcal{L}_r(y) = 0$ , and it is the  $y_r^m$  given in part (a).

To prove part (b), from Equation (3), we have

$$\begin{aligned} \frac{\partial}{\partial y} \mathcal{L}_s(y) &= (1 - \alpha)c_s + \alpha^L h_s \\ &\quad + \alpha^{L+1} \frac{\partial}{\partial y} \mathbb{E}[\text{IP}^m(y - D^{L+1})]. \end{aligned} \quad (12)$$

From Equation (2), we have

$$\frac{\partial}{\partial y} \text{IP}^m(y) = \begin{cases} \frac{\partial}{\partial y} \mathcal{L}_r(y), & y \leq y_r^m, \\ 0, & y > y_r^m. \end{cases}$$

By the definition of  $y_r^m$ , the above function is continuous over all  $x$ . We define the first derivative of  $\mathcal{L}_s(y)$  in two regions.

When  $y \leq y_r^m$ , then  $y - u \leq y_r^m$  for all  $u$ . Hence, from Equation (12), we have

$$\begin{aligned} \frac{\partial}{\partial y} \mathcal{L}_s(y) &= (1 - \alpha)c_s + \alpha^L h_s + \alpha^{L+1} \int_0^\infty \frac{\partial}{\partial y} \mathcal{L}_r(y - u) f_{L+1}(u) du \\ &= (1 - \alpha)c_s + \alpha^L h_s + \alpha^{L+1} [(1 - \alpha)c_r - \alpha^l(p_r + h_s)] \\ &\quad + \alpha^{L+1+1}(p_r + h_r) F_{L+1+2}(y), \end{aligned} \quad (13)$$

which is increasing in  $y$ .

When  $y > y_r^m$ , we have  $(\partial/\partial y)\text{IP}^m(y - u) = 0$  for  $u < y - y_r^m$ . Hence, from Equation (12), we have

$$\begin{aligned} \frac{\partial}{\partial y} \mathcal{L}_s(y) &= (1 - \alpha)c_s + \alpha^L h_s + \alpha^{L+1} \int_{y-y_r^m}^\infty \frac{\partial}{\partial y} \mathcal{L}_r(y - u) f_{L+1}(u) du \\ &= (1 - \alpha)c_s + \alpha^L h_s + \alpha^{L+1} [(1 - \alpha)c_r - \alpha^l(p_r + h_s)] \\ &\quad \cdot [1 - F_{L+1}(y - y_r^m)] + \alpha^{L+1+1}(p_r + h_r) \\ &\quad \cdot \int_{y-y_r^m}^\infty F_{l+1}(y - u) f_{L+1}(u) du, \end{aligned} \quad (14)$$

which is increasing in  $y$  over the region  $y \geq y_r^m$  because  $(\partial^2/\partial y^2)\mathcal{L}_s(y) = \alpha^{L+1+1}(p_r + h_r) \int_{y-y_r^m}^\infty f_{l+1}(y - u) f_{L+1}(u) du > 0$ . Because it is also increasing in the other region,  $(\partial/\partial y)\mathcal{L}_s(y)$  is increasing in  $y$  for all  $y$ .

Because  $p_r \geq P$ , we have  $(\partial/\partial y)\mathcal{L}_s(0) = (1 - \alpha)c_s + \alpha^L h_s + \alpha^{L+1} [(1 - \alpha)c_r - \alpha^l(p_r + h_s)] \leq 0$ , and  $\lim_{y \rightarrow \infty} (\partial/\partial y)\mathcal{L}_s(y) = (1 - \alpha)c_s + \alpha^L h_s > 0$ . Hence,  $(\partial/\partial y)\mathcal{L}_s(y)$  crosses the zero line only once, and  $y_s^m$  is defined as the unique point where it crosses zero.

When  $p_r \leq H$ , we have  $(\partial/\partial y)\mathcal{L}_s(y_r^m) \geq 0$ . Therefore,  $y_s^m \in [0, y_r^m]$  and it is obtained by setting Equation (13) equal to zero; i.e.,  $(\partial/\partial y)\mathcal{L}_s(y_s^m) = 0$ . Solving for  $y_s^m$  yields the closed-form solution in part (b). Note that  $y_s^m \leq y_r^m$ .

When  $p_r > H$ , we have  $(\partial/\partial y)\mathcal{L}_s(y_r^m) < 0$ . Hence,  $y_s^m \in (y_r^m, \infty)$  and it is the  $y$  that sets Equation (14) equal to zero. Note also that  $y_s^m > y_r^m$ .

The installation base-stock levels are obtained from their definition; i.e.,  $Y_r = \min\{y_s^m, y_r^m\}$  and  $Y_s = \max\{y_s^m - y_r^m, 0\}$ .

**PROOF OF PROPOSITION 9.** To prove part (a), note that  $Y_r = y_r^m$  when  $p_r > H$ . We set  $Y_r = F_{l+1}^{-1}((\alpha^l(p_r + h_s) - (1 - \alpha)c_r)/\alpha^l(p_r + h_r)) = F_{l+1+\tau}^{-1}((p_r - (1 - \alpha)c_r)/(p_r + h_r)) = Y_r^*(\tau)$ . Applying  $F_{l+1}(\cdot)$  to both sides and rearranging terms yields the threshold  $B(\tau)$ . Note that when  $h_s \leq B(\tau)$ ,  $Y_r \leq Y_r^*(p_r, \tau)$ . Note also that  $B(\tau)$  is increasing in  $\tau$  because  $F_{l+1+\tau}(x) \geq F_{l+1+\tau'}(x)$  for any  $x \geq 0$  and  $\tau \leq \tau'$ .

Part (b) follows immediately from Proposition 8(b) and the definition of  $Y_s^*(\tau)$ .  $\square$

**PROOF OF PROPOSITION 10.** The proof follows from concavity and Proposition 2(a) and 2(b).  $\square$

**PROOF OF PROPOSITION 11.** First, recall from Proposition 5(c) that  $\tau_1^a = \tau_1^f$ . Hence,  $\tau_1^a = \{0, L + 1\}$  because  $G_s^*(\tau) + G_r^*(p_1, \tau)$  is concave. Second, consider the two trivial cases. Note that  $L + 1 \geq \tau_1^a \geq \tau_2^a \geq \dots \geq \tau_N^a \geq 0$  from Proposition 5(a). Hence, if  $\tau_1^a = 0$ , then  $\tau_i^a = 0$  for all  $i$ . Similarly, if  $\tau_N^a = L + 1$ , then  $\tau_i^a = L + 1$  for all  $i$ . Parts (a) and (b) and the expression for  $K_i^a$  follow from Proposition 4(a). Next, consider  $\tau_1^a = L + 1$  and  $\tau_N^a \neq L + 1$ .

Note that each term in the objective function (6) is concave in  $\tau$ , that is,  $W_i(\tau) \equiv \lambda_i[G_s^*(\tau) + G_r^*(p_i, \tau)] + \bar{\lambda}_{i-1}[G_r^*(p_i, \tau) - G_r^*(p_{i-1}, \tau)]$  is concave in  $\tau$ . Let  $\tau_i^{\max}$  be the smallest maximizer.

Next, we show that  $\tau_i^a \in \{\tau_{i-1}^a, \tau_{i+1}^a\}$  for  $i \in \{2, \dots, N - 1\}$ . Assume for a contradiction that  $\tau_{i-1}^a > \tau_i^a > \tau_{i+1}^a$ . We consider three cases:  $\tau_i^{\max} < \tau_i^a$ ,  $\tau_i^{\max} > \tau_i^a$ , and  $\tau_i^{\max} = \tau_i^a$ . If  $\tau_i^{\max} < \tau_i^a$ , then  $(\partial/\partial \tau)W_i(\tau) < 0$  for all  $\tau \in [\tau_i^a, \tau_{i-1}^a]$ . Therefore, the supplier can increase  $\tau_i^a$  to  $\tau_{i-1}^a$  and reduce her expected cost without violating the monotonicity constraint in (6), contradicting the optimality of  $\tau_i^a$ . If  $\tau_i^{\max} > \tau_i^a$ , then  $(\partial/\partial \tau)W_i(\tau) > 0$  for all  $\tau \in (\tau_{i+1}^a, \tau_i^a]$ . Therefore, the supplier could decrease  $\tau_i^a$  to  $\tau_{i+1}^a$  and reduce her expected cost without violating the monotonicity constraint in (6), contradicting the optimality of  $\tau_i^a$ . If  $\tau_i^{\max} = \tau_i^a$ , then  $\tau_i \in \{\tau_{i+1}^a, \tau_{i-1}^a\}$  yields a lower expected cost for the supplier without violating monotonicity, again contradicting the optimality of  $\tau_i^a$ . Therefore, for all  $i \in \{2, \dots, N - 1\}$ , we must have  $\tau_i^a \in \{\tau_{i-1}^a, \tau_{i+1}^a\}$ . Similar arguments lead to  $\tau_N^a \in \{\tau_{N-1}^a, 0\}$ .

The above result implies that there exists a finite strictly increasing sequence  $k_1, k_2, \dots, k_n$  (where  $k_n \equiv N$ ) for some  $n \geq 0$  such that  $\tau_1^a = \tau_2^a = \dots = \tau_{k_1}^a = L + 1$ ;  $\tau_{k_1+1}^a = \dots = \tau_{k_2}^a \equiv \tau_{k_2}$ ;  $\dots$  and  $\tau_{k_{n-1}+1}^a = \tau_{k_n}^a = \dots = \tau_{k_n}^a \equiv \tau_{k_n}$ . In other words, there are  $n$  segments. Next, we show that in fact  $n \leq 2$ , i.e., there can be at most two such segments.

Consider the segment  $k_j$  for some  $j \in \{2, \dots, N - 1\}$ . Note that  $W_{k_j}(\tau_{k_j}) \equiv \sum_{k=k_{j-1}+1}^{k_j} \lambda_k[G_s^*(\tau_{k_j}) + G_r^*(p_k, \tau_{k_j})] +$

$\bar{\lambda}_{k-1} [G_r^*(p_k, \tau_{k_j}) - G_r^*(p_{k-1}, \tau_{k_j})]$  is also concave in  $\tau_{k_j}$ . Let  $\tau_{k_j}^{\max}$  be the maximizer of  $W_{k_j}(\tau_{k_j})$ . Following the same arguments as above, one can show that  $\tau_{k_j} \in \{\tau_{k_{j-1}}, \tau_{k_{j+1}}\}$ . Hence, the  $n$  segments can be reduced to  $n - 1$  segments, that is, there exists a new finite strictly increasing sequence  $k'_1, k'_2, \dots, k'_{n-1}$ . This argument can be applied iteratively until the sequence of numbers is reduced to two numbers, and hence two segments. Therefore, there exists an  $m$  such that  $\tau_i^a = \tau_1^a = L + 1$  for  $i \leq m$  and  $\tau_i^a = \tau_N^a \neq 0$  for  $i > m$ . The expression for  $K_i^a$  follows from Proposition 4(a).  $\square$

PROOF OF PROPOSITION 12. Let  $G^*(n)$  be a general form of the optimal expected cost function, where  $n$  is the number of periods of uncertainty, i.e.,  $n = L + 1 - \tau$  for the supplier or  $n = l + 1 + \tau$  for the retailer. Then, from (9),  $G^*(n) = p n \mu - (h + p) \int_0^{F_n^{-1}(p/(h+p))} u f_n(u) du$  for generic holding and penalty costs  $h$  and  $p$ .

To prove part (i), when  $F(\cdot)$  is Normally distributed, the optimal base-stock level is  $Y^* = F_n^{-1}(p/(h + p)) = n\mu + z\sigma\sqrt{n}$ , where  $z = \Phi^{-1}(p/(h + p))$ . The resulting optimal cost is  $G^*(n) = (h + p)\phi(z)\sigma\sqrt{n}$ . Hence,  $G^*(n)$  depends on  $\tau$  only through  $\sqrt{n}$ , which is concave in  $\tau$  for both  $n = L + 1 - \tau$  and  $n = l + 1 + \tau$ . Note also that we have  $G_r^*(p_i, \tau) - G_r^*(p_{i-1}, \tau) = [(h_r + p_i)\phi(\Phi(p_i/(h_r + p_i)))\sigma - (h_r + p_{i-1})\phi(\Phi(p_{i-1}/(h_r + p_{i-1})))\sigma]\sqrt{l + 1 + \tau}$ . The term in brackets is the difference between optimal inventory cost functions with  $l = \tau = 0$  and penalty costs  $p_i$  and  $p_{i-1}$ , respectively. From Proposition 1(b), this term is positive. Hence,  $G_r^*(p_i, \tau) - G_r^*(p_{i-1}, \tau)$  is concave in  $\tau$ .

To prove part (ii), we define  $\Delta_n G^*(n) = G^*(n) - G^*(n - 1)$ . To show that  $G^*(n)$  is concave in  $n$ , we show  $\Delta_n G^*(n) > \Delta_{n+1} G^*(n)$ . By algebra, this inequality holds when

$$\int_0^{F_{n+1}^{-1}(p/(h+p))} u f_{n+1}(u) du - 2 \int_0^{F_n^{-1}(p/(h+p))} u f_n(u) du + \int_0^{F_{n-1}^{-1}(p/(h+p))} u f_{n-1}(u) du > 0. \tag{15}$$

Letting  $r = p/(h + p)$ , we note that the left side of (15) equals zero for  $r \in \{0, 1\}$ . The second derivative of the left side of (15) with respect to  $r$  equals  $1/f_{n+1}(F_{n+1}^{-1}(r)) - 2/f_n(F_n^{-1}(r)) + 1/f_{n-1}(F_{n-1}^{-1}(r))$ , which must be negative for (15) to be true for all  $r \in (0, 1)$ . That is,  $[f_n(F_n^{-1}(r))]^{-1}$  must be concave in  $n$ .

Next, we show that  $G_r^*(p_i, \tau) - G_r^*(p_{i-1}, \tau)$  is concave in  $\tau$ . To do so, we show that  $(\partial/\partial p_r)[G_r^*(p_r, \tau) - 2G_r^*(p_r, \tau - 1) + G_r^*(p_r, \tau - 2)] \leq 0$ . From (9), the above is equivalent to

$$\left[ \int_0^{F_{l+1+\tau}^{-1}(p_r/(h_r+p_r))} u f_{l+1+\tau}(u) du - 2 \int_0^{F_{l+\tau}^{-1}(p_r/(h_r+p_r))} u f_{l+\tau}(u) du + \int_0^{F_{l+\tau-1}^{-1}(p_r/(h_r+p_r))} u f_{l+\tau-1}(u) du \right] + \left( \frac{h_r}{h_r + p_r} \right)$$

$$\left[ F_{l+1+\tau}^{-1} \left( \frac{p_r}{h_r + p_r} \right) - 2F_{l+\tau}^{-1} \left( \frac{p_r}{h_r + p_r} \right) + F_{l+\tau-1}^{-1} \left( \frac{p_r}{h_r + p_r} \right) \right] \geq 0. \tag{16}$$

The limit of the left side of (16) as  $r = p_r/(h_r + p_r)$  goes to one equals  $(l + 1 + \tau)\mu - 2(l + \tau)\mu + (l + \tau - 1)\mu = 0$ . Hence, (16) holds as long as the derivative of the left side with respect to  $r$  is negative. This derivative equals  $(1 - r)[1/f_{l+1+\tau}(F_{l+1+\tau}^{-1}(r)) - 2/f_{l+\tau}(F_{l+\tau}^{-1}(r)) + 1/f_{l+\tau-1}(F_{l+\tau-1}^{-1}(r))]$ , which is negative when  $[f_n(F_n^{-1}(r))]^{-1}$  is concave in  $n$ .  $\square$

PROOF OF PROPOSITION 13. The proof is deferred to the online companion that can be found at <http://orjournal.informs.org/>.  $\square$

### Endnotes

1. Recent information technologies such as compliance management systems and RFID have been cited as enablers for accounting of service- and quality-related activities across the supply chain (Lee and Özer 2005).
2. In §5, we show that the retailer follows a stationary base-stock policy, so the retailer orders in each period to recover the previous period’s customer demand. As a result, the supplier observes the same demand stream as the retailer.
3. We drop the subscript  $t$  from the definition of  $D^{n+1}$  for brevity.
4. This alternative source may be a finished-goods inventory carried either by another firm or for another retailer. It may also represent production or processing resources lent by another operation. To our knowledge, this alternative source concept appeared first in Lee et al. (2000). See also Graves and Willems (2000), who assume guaranteed service between every stage in a supply chain.
5. We note the difference between the *centralized* (first-best) solution concept used in supply chain contracting literature and the *central* control concepts used in multiechelon inventory literature. In the literature, these nomenclatures are rarely discussed together. This is due to the fact that most papers either study coordination issues in supply chain contracting for a single-period problem (as in Cachon 2003), or they study central and local control in multiechelon inventory problems under full information (as in Axsäter and Rosling 1993).
6. Inventory at firm  $j$  and downstream plus the pipeline inventory to firm  $j$  minus customer backorders.
7. Inventory on hand at firm  $j$  plus its pipeline inventory minus the backorder due to downstream location’s order (or customer demand for  $j = r$ ).
8. Using a similar argument in Veinott (1965), the inventory control problem with linear salvage value for each firm is converted into an equivalent problem with zero salvage.
9. We use the term decreasing and increasing in the weak sense, so decreasing means nonincreasing.

## Acknowledgments

The authors are thankful to the anonymous associate editor and the referees for their helpful suggestions. The discussions during their presentations at the 2003 INFORMS annual meeting in Atlanta, Stanford University, Texas A&M, University of California–Santa Cruz, University of Illinois–Urbana-Champaign, University of Tennessee–Knoxville, University of Texas–Dallas, L. M. Ericsson, and Hewlett-Packard were also beneficial. The authors also gratefully acknowledge financial support from L. M. Ericsson and the Stanford Alliance for Innovative Manufacturing. The previous version of this paper was titled “Promised Leadtime Contracts and Renegotiation Incentives Under Asymmetric Information.” The listing of authors is alphabetical.

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