Online Companion for

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Proof of Lemma 1. Assume for a contradiction that $x_f < x_g$, we have $0 \leq \nabla f(x_g - 1) \leq \nabla g(x_g - 1)$. The first inequality is due to the assumption $x_f < x_g$; the second one is due to the statement $\nabla f(x) \leq \nabla g(x)$. Then $\nabla g(x_g - 1) \geq 0$ but this contradicts the definition of x_g as the smallest minimizer. Hence $x_f \geq x_g$. □

Proof of Lemma 2. For all x and t,

$$\begin{aligned} \nabla V_t(x-\epsilon, o) &= \nabla G_t(x-\epsilon) + \alpha E \nabla J_{t+1}(x - \epsilon - o_{t,t+L+1} - \sum_{s=t}^{t+L+1} D_{t,s}, O_{t+1}) \\ &\leq \nabla G_t(x) + \alpha E \nabla J_{t+1}(x - \epsilon - o_{t,t+L+1} - \sum_{s=t}^{t+L+1} D_{t,s}, O_{t+1}) \\ &= \nabla V_t(x, o + \epsilon e_1). \end{aligned} \tag{1}$$

The inequality is due to the convexity of $G_t(x)$. Notice that the smallest minimizer of $V_t(x - \epsilon, o)$ is nothing but $y_t(o) + \epsilon$. This together with Lemma 1 implies $y_t(o + \epsilon e_1) \leq y_t(o) + \epsilon$. From equation (8) in Özer and Wei [1], we have

$$\nabla J_t(x, o) \begin{cases} < 0, & x < y_t(o) - Q \\ = 0, & y_t(o) - Q \leq x < y_t(o) \\ \geq 0, & y_t(o) \leq x \end{cases} \tag{2}$$

Next we prove $\nabla J_t(x - \epsilon, o) \leq \nabla J_t(x, o + \epsilon e_1)$ for all x and t. To do so, we use the above equation (2) and equation (8) in Özer and Wei [1], and consider 9 cases:

Case 1: $x - \epsilon < y_t(o) - Q$ and $x < y_t(o + \epsilon e_1) - Q$ then $\nabla J_t(x - \epsilon, o) = \nabla V_t(x - \epsilon + Q, o) \leq \nabla V_t(x + Q, o + \epsilon e_1) = \nabla J_t(x, o + \epsilon e_1)$. The inequality follows from equation (1).

Case 2: $x - \epsilon < y_t(o) - Q$ and $y_t(o + \epsilon e_1) - Q \leq x < y_t(o + \epsilon e_1)$ then $\nabla J_t(x - \epsilon, o) < 0 = \nabla J_t(x, o + \epsilon e_1)$.

Case 3: $x - \epsilon < y_t(o) - Q$ and $y_t(o + \epsilon e_1) \leq x$ then $\nabla J_t(x - \epsilon, o) < 0 \leq \nabla J_t(x, o + \epsilon e_1)$.

Case 4: $y_t(o) - Q \leq x - \epsilon < y_t(o)$ and $x < y_t(o + \epsilon e_1) - Q$ is not possible, since $x < y_t(o + \epsilon e_1) - Q \leq y_t(o) + \epsilon - Q$.

Case 5: $y_t(o) - Q \leq x - \epsilon < y_t(o)$ and $y_t(o + \epsilon e_1) - Q \leq x < y_t(o + \epsilon e_1)$ then $\nabla J_t(x - \epsilon, o) = 0 = \nabla J_t(x, o + \epsilon e_1)$.

Case 6: $y_t(o) - Q \leq x - \epsilon < y_t(o)$ and $y_t(o + \epsilon e_1) \leq x$ then $\nabla J_t(x - \epsilon, o) = 0 \leq \nabla J_t(x, o + \epsilon e_1)$.

Case 7: $y_t(o) \leq x - \epsilon$ and $x < y_t(o + \epsilon e_1) - Q$ is not possible as in Case 4.

Case 8: $y_t(o) \leq x - \epsilon$ and $y_t(o + \epsilon e_1) - Q \leq x < y_t(o + \epsilon e_1)$ is not possible as in Case 4.

Case 9: $y_t(o) \leq x - \epsilon$ and $y_t(o + \epsilon e_1) \leq x$ then $\nabla J_t(x - \epsilon, o) = \nabla V_t(x - \epsilon, o) \leq \nabla V_t(x, o + \epsilon e_1) = \nabla J_t(x, o + \epsilon e_1)$. The inequality follows from equation (1). □

Proof of Lemma 3. Since $H_t(x, o)$ is increasing in x (Theorem 5) , the first part follows immediately from equation (6) in Özer and Wei [1]. The second part is shown in the proof for Theorem 5. □

Proof of Lemma 4. For all t and any o, we have

$$\begin{aligned}
H_t(x-\epsilon, o) &= K_t + G_t(x - \epsilon + Q) + \alpha E\left[J_{t+1}(x - \epsilon + Q - o_{t,t+L+1} - \sum_{s=t}^{t+L+1} D_{t,s}, O_{t+1})\right] \\
&\quad -G_t(x-\epsilon) - E\left[J_{t+1}(x - \epsilon - o_{t,t+L+1} - \sum_{s=t}^{t+L+1} D_{t,s}, O_{t+1})\right] \\
&\leq K_t + G_t(x+Q) - G_t(x) + \alpha E\left[J_{t+1}(x - \epsilon + Q - o_{t,t+L+1} - \sum_{s=t}^{t+L+1} D_{t,s}, O_{t+1})\right. \\
&\quad \left. -J_{t+1}(x - \epsilon - o_{t,t+L+1} - \sum_{s=t}^{t+L+1} D_{t,s}, O_{t+1})\right] \\
&= H_t(x, o + \epsilon e_1)
\end{aligned}$$

The inequality is due to the convexity of $G_t(x)$. From equation (7) in Özer and Wei [1], $\min\{x : H_t(x - \epsilon, o) \geq 0\} = s_t(o) + \epsilon$. Because $H_t(x, o)$ is increasing, we have $s_t(o + \epsilon e_1) \leq s_t(o) + \epsilon$. From equation (10) in Özer and Wei [1], we have

$$J_t(x+Q, o) - J_t(x, o) = \begin{cases} H_t(x+Q, o) - K_t < -K_t, & x < s_t(o) - Q \\ -K_t, & s_t(o) - Q \leq x < s_t(o) \\ H_t(x, o) - K_t \geq -K_t, & s_t(o) \leq x \end{cases} \tag{3}$$

Next we show that $J_t(x - \epsilon + Q, o) - J_t(x - \epsilon, o) \leq J_t(x + Q, o + \epsilon e_1) - J_t(x, o + \epsilon e_1)$. To do so, we use equation (3) and consider 9 cases:

Case 1: $x - \epsilon < s_t(o) - Q$ and $x < s_t(o + \epsilon e_1) - Q$ then $J_t(x - \epsilon + Q, o) - J_t(x - \epsilon, o) = H_t(x - \epsilon + Q, o) - K_t \leq H_t(x + Q, o + \epsilon e_1) - K_t = J_t(x + Q, o + \epsilon e_1) - J_t(x, o + \epsilon e_1)$.

Case 2: $x - \epsilon < s_t(o) - Q$ and $s_t(o + \epsilon e_1) - Q \leq x < s_t(o + \epsilon e_1)$ then $J_t(x - \epsilon + Q, o) - J_t(x - \epsilon, o) = H_t(x - \epsilon + Q, o) - K_t < -K_t = J_t(x + Q, o + \epsilon e_1) - J_t(x, o + \epsilon e_1)$.

Case 3: $x - \epsilon < s_t(o) - Q$ and $s_t(o + \epsilon e_1) \leq x$ then $J_t(x - \epsilon + Q, o) - J_t(x - \epsilon, o) = \underbrace{H_t(x - \epsilon + Q, o)}_{<0} - K_t < \underbrace{H_t(x, o + \epsilon e_1)}_{\geq 0} - K_t = J_t(x + Q, o + \epsilon e_1) - J_t(x, o + \epsilon e_1)$.

Case 4: $s_t(o) - Q \leq x - \epsilon < s_t(o)$ and $x < s_t(o + \epsilon e_1) - Q$ is not possible because $x + Q < s_t(o + \epsilon e_1) \leq s_t(o) + \epsilon$.

Case 5: $s_t(o) - Q \leq x - \epsilon < s_t(o)$ and $s_t(o + \epsilon e_1) - Q \leq x < s_t(o + \epsilon e_1)$ then $J_t(x - \epsilon + Q, o) - J_t(x - \epsilon, o) = -K_t = J_t(x + Q, o + \epsilon e_1) - J_t(x, o + \epsilon e_1)$.

Case 6: $s_t(o) - Q \leq x - \epsilon < s_t(o)$ and $s_t(o + \epsilon e_1) \leq x$ then $J_t(x - \epsilon + Q, o) - J_t(x - \epsilon, o) = -K_t \leq H_t(x, o + \epsilon e_1) - K_t = J_t(x + Q, o + \epsilon e_1) - J_t(x, o + \epsilon e_1)$.

Case 7: $s_t(o) \leq x - \epsilon$ and $x < s_t(o + \epsilon e_1) - Q$ is not possible as in Case 4.

Case 8: $s_t(o) \leq x - \epsilon$ and $s_t(o + \epsilon e_1) - Q \leq x < s_t(o + \epsilon e_1)$ is not possible as in Case 4.

Case 9: $s_t(o) \leq x - \epsilon$ and $s_t(o + \epsilon e_1) \leq x$ then $J_t(x - \epsilon + Q, o) - J_t(x - \epsilon, o) = H_t(x - \epsilon, o) - K_t \leq H_t(x, o + \epsilon e_1) - K_t = J_t(x + Q, o + \epsilon e_1) - J_t(x, o + \epsilon e_1)$. □

Proof of Theorem 3. For Part 1, we show that $\nabla V_t(x, o|Q_2) \geq \nabla V_t(x, o|Q_1)$ for all t. For $t = T$, it holds as an equality since $\nabla V_T(x, o)$ does not depend on Q. Assume for an induction argument that the inequality is true for t, then $y_t(o|Q_2) \leq y_t(o|Q_1)$ (Part 1 for t) follows from Lemma 1. Next we show that this implies $\nabla J_t(x, o|Q_2) \geq \nabla J_t(x, o|Q_1)$. To do so, we use equation (2) and the induction argument and consider 3 cases:

Case 1: $x < y_t(o|Q_2) - Q_2$ then $\nabla J_t(x, o|Q_2) = \nabla V_t(x + Q_2, o|Q_2) \geq \nabla V_t(x + Q_2, o|Q_1) \geq \nabla V_t(x + Q_1, o|Q_1) = \nabla J_t(x, o|Q_1)$. The second inequality is due to the convexity of $V_t(x, o)$.

Case 2: $y_t(o|Q_2) - Q_2 \leq x < y_t(o|Q_1)$ then $\nabla J_t(x, o|Q_2) \geq 0 \geq \nabla J_t(x, o|Q_1)$.

Case 3: $y_t(o|Q_1) \leq x$ then $\nabla J_t(x, o|Q_2) = \nabla V_t(x, o|Q_2) \geq \nabla V_t(x, o|Q_1) = \nabla J_t(x, o|Q_1)$.

The three cases above and equation (4) in Özer and Wei [1] imply that $\nabla V_{t-1}(x, o|Q_2) \geq \nabla V_{t-1}(x, o|Q_1)$, concluding the induction argument and the proof for Part 1.

For Part 2, note that increasing the capacity limit is equivalent to relaxing the constraint set in the dynamic program. Hence the result follows immediately. □

Proof of Theorem 7. The proof for Part 1 is a simple induction. Since $K_T^2 > K_T^1$, $J_T(x, o|K^2) = \min\{K_T^2 + G_T(x + Q), G_T(x)\} \geq \min\{K_T^1 + G_T(x + Q), G_T(x)\} = J_T(x, o|K^1)$.

Hence it is true for $t = T$. Assume it is true for t, then $V_{t-1}(x, o|K^2) = G_{t-1}(x) + \alpha E J_t(x', O|K^2) \geq G_{t-1}(x) + \alpha E J_t(x', O|K^1) = V_{t-1}(x, o|K^1)$. Consequently, $J_{t-1}(x, o|K^2) = \min\{K_{t-1}^2 + V_{t-1}(x + Q|K^2), V_{t-1}(x|K^2)\} \geq \min\{K_{t-1}^1 + V_{t-1}(x + Q|K^1), V_{t-1}(x|K^1)\} = J_{t-1}(x, o|K^1)$.

We prove Part 2 and 3 by induction. For $t = T$, $H_T(x, o|K^2) = K_T^2 + G_T(x+Q) - G_T(x) > K_T^1 + G_T(x + Q) - G_T(x) = H_T(x, o|K^1)$. Hence Part 2 is true for $t = T$. Now assume that Part 2 holds for t, Part 3 for t follows immediately because $H_t(x, o)$ is increasing in x. Next we show that $K_t^2 + J_t(x + Q, o|K^2) - J_t(x, o|K^2) \geq K_t^1 + J_t(x + Q, o|K^1) - J_t(x, o|K^1)$. To do so, we use equation (3) and the induction argument and consider 9 cases:

Case 1: $x < s_t(o|K^2) - Q$ and $x < s_t(o|K^1) - Q$ then $K_t^2 + J_t(x+Q, o|K^2) - J_t(x, o|K^2) = H_t(x + Q, o|K^2) \geq H_t(x + Q, o|K^1) = K_t^1 + J_t(x + Q, o|K^1) - J_t(x, o|K^1)$.

Case 2: $x < s_t(o|K^2) - Q$ and $s_t(o|K^1) - Q \leq x < s_t(o|K^1)$ is not possible since $s_t(o|K^2) \leq s_t(o|K^1)$.

Case 3: $x < s_t(o|K^2) - Q$ and $s_t(o|K^1) \leq x$ is not possible since $s_t(o|K^2) \leq s_t(o|K^1)$.

Case 4: $s_t(o|K^2) - Q \leq x < s_t(o|K^2)$ and $x < s_t(o|K^1) - Q$ then $K_t^2 + J_t(x + Q, o|K^2) - J_t(x, o|K^2) = 0 > H_t(x + Q, o|K^1) = K_t^1 + J_t(x + Q, o|K^1) - J_t(x, o|K^1)$.

Case 5: $s_t(o|K^2) - Q \leq x < s_t(o|K^2)$ and $s_t(o|K^1) - Q \leq x < s_t(o|K^1)$ then $K_t^2 + J_t(x + Q, o|K_t^2) - J_t(x, o|K_t^2) = 0 = K_t^1 + J_t(x + Q, o|K^1) - J_t(x, o|K^1)$.

Case 6: $s_t(o|K^2) - Q \leq x < s_t(o|K^2)$ and $s_t(o|K^1) \leq x$ is not possible.

Case 7: $s_t(o|K^2) \leq x$ and $x < s_t(o|K^1) - Q$ then $K_t^2 + J_t(x + Q, o|K^2) - J_t(x, o|K^2) = H_t(x, o|K^2) \geq 0 > H_t(x + Q, o|K^1) = K_t^1 + J_t(x + Q, o|K^1) - J_t(x, o|K^1)$.

Case 8: $s_t(o|K^2) \leq x$ and $s_t(o|K^1) - Q \leq x < s_t(o|K^1)$ then $K_t^2 + J_t(x + Q, o|K^2) - J_t(x, o|K^2) = H_t(x, o|K^2) \geq 0 = K_t^1 + J_t(x + Q, o|K^1) - J_t(x, o|K^1)$.

Case 9: $s_t(o|K^2) \leq x$ and $s_t(o|K^1) \leq x$ then $K_t^2 + J_t(x + Q, o|K^2) - J_t(x, o|K^2) = H_t(x, o|K^2) \geq H_t(x, o|K^1) = K_t^1 + J_t(x + Q, o|K^1) - J_t(x, o|K^1)$.

We complete the induction argument by showing Part 2 for $t - 1$. From equation (4) and (6) in Özer and Wei [1], we have $H_{t-1}(x, o|K^2) = K_{t-1}^2 - \alpha K_t^2 + G_{t-1}(x + Q) - G_{t-1}(x) + \alpha E[K_t^2 + J_t(x' + Q, O|K^2) - J_t(x', O|K^2)] \geq K_{t-1}^1 - \alpha K_t^1 + G_{t-1}(x + Q) - G_{t-1}(x) + \alpha E[K_t^1 +$

$J_t(x'+Q,O|K^1) - J_t(x',O|K^1)] = H_{t-1}(x,o|K_1)$ The inequality is due to the condition and the above 9 cases. □

Proof of Theorem 8. For Part 1 and 2, we first prove that for any given o, $J_t(x+Q,o) - J_t(x,o) \leq J_{t+1}(x+Q,o) - J_{t+1}(x,o)$ for $x \leq s_t(o)$. For $t = T$, we prove $J_T(x+Q,o) - J_T(x,o) < 0 = J_{T+1}(x+Q,o) - J_{T+1}(x,o)$ for $x \leq s_T(o)$ by using equation (3) and considering 2 cases:

Case 1: $x < s_T(o) - Q$ then $J_T(x+Q,o) - J_T(x,o) = H_T(x+Q,o) - K < 0$.

Case 2: $s_T(o) - Q \leq x \leq s_T(o)$ then $J_T(x+Q,o) - J_T(x,o) = -K < 0$.

Assume for an induction argument that this inequality holds for t. We can show that $H_{t-1}(x,o) \leq H_t(x,o)$ for $x \leq s_t(o)$ through an argument similar to that of Theorem 4. Because $H_{t-1}(x,o)$ is increasing, $s_{t-1}(o) \geq s_t(o)$ follows from the definition (equation (7) in Özer and Wei [1]). We complete the induction argument by considering 2 cases:

Case 1: $x < s_t(o) - Q$ then $J_{t-1}(x+Q,o) - J_{t-1}(x,o) = H_{t-1}(x+Q,o) - K \leq H_t(x+Q,o) - K = J_t(x+Q,o) - J_t(x,o)$.

Case 2: $s_t(o) - Q \leq x \leq s_{t-1}(o)$ then from equation (3) we have $J_{t-1}(x+Q,o) - J_{t-1}(x,o) \leq -K \leq J_t(x+Q,o) - J_t(x,o)$.

For Part 3, note that since $G(x)$ is nonnegative, $J_T(x,o) = \min\{K + G(x+Q), G(x)\} \geq 0 = J_{T+1}(x,o)$. Hence Part 3 is true for $t = T$. Assume it is true for t, then $V_{t-1}(x,o) = G(x) + \alpha E J_t(x',O) \geq G(x) + \alpha E J_{t+1}(x',O) = V_t(x,o)$. Consequently, $J_{t-1}(x,o) = \min\{K + V_{t-1}(x+Q), V_{t-1}(x)\} \geq \min\{K + V_t(x+Q), V_t(x)\} = J_t(x,o)$. □

Proof of Theorem 9. For any policy Y we define the total expected cost for the $(T-t)$-period problem as $B_t(x,o|T,Y) = E\sum_{j=t}^{T} \alpha^{j-t} G_j(y_j)$, and the total expected cost for the infinite horizon problem as $B(x,o|Y) = E\lim_{T\to\infty} \sum_{j=t}^{T} \alpha^{j-t} G_j(y_j)$. By the monotone convergence theorem, $\lim_{T\to\infty} B_t(x,o|T,Y) = B(x,o|Y)$. Let Y^* be an optimal policy for $(T-t)$-period problem, then $J_t(x,o|T) = B_t(x,o|T,Y^*) \leq B_t(x,o|T,Y) \leq B(x,o|Y)$. Using a similar argument in Federgruen and Zipkin [2] (pg 210, Theorem 1), we can show that $B(x,o|Y) = O(|x|^p)$. Hence $\{J_t(x,o|T)\}$ is bounded above. From Theorem 4 $\{J_t(x,o|T)\}$ is increasing in T. Therefore, it converges to a limit $J(x,o)$ which inherits convexity and coerciveness from $J_t(x,o|T)$, completing the proof for Part 1.

For Part 2, since $EJ(x - o^1 - Z, O) < \infty$ for all x, $V(x, o)$ is well defined. The convexity and coerciveness of V follows from those of G and J, so $y_\infty(o)$ is finite. Since each $J_t(x, o|T) \leq J(x, o)$, by the Lebesgue Convergence Theorem, $EJ_t(x - o^1 - Z, O|T) \to EJ(x - o^1 - Z, O)$, so $V_t(x, o|T) \to V(x, o)$. To prove that $\lim_{T\to\infty} y_t(o|T) = y_\infty(o)$, we define $\bar{y}(o) = \sup\{y_t(o|T)\}_{T=t}^{\infty}$ and use contradiction. If $y_\infty(o) < \bar{y}(o)$, then choose N such that $y_\infty(o) < y_t(o|N) < \bar{y}(o)$. For $T \geq N$,

$$V_t(y_\infty(o), o|T) - V_t(y_t(o|N), o|T) = -\sum_{y_\infty(o)}^{y_t(o|N)-1} \nabla V_t(y, o|T) \geq -\sum_{y_\infty(o)}^{y_t(o|N)-1} \nabla V_t(y, o|N) > 0,$$

the first inequality is due to Theorem 4 Part 1 and the second inequality is a result of our choice for N. Taking the limit of the first term we have $V(y_\infty(o), o) > V(y_t(o|N), o)$, contradicting the optimality of $y_\infty(o)$. On the other hand if $y_\infty(o) > \bar{y}(o)$, then since $V_t(y, o|T)$ is increasing for $y \geq \bar{y}(o) \geq y_t(o|T)$ for all $T \geq t$, we have $V_t(\bar{y}(o), o|T) \leq V_t(y_\infty(o), o|T) \leq V(y_\infty(o), o)$. Therefore, $V(\bar{y}(o), o) - V_t(\bar{y}(o), o|T) \geq V(\bar{y}(o), o) - V(y_\infty(o), o) > 0$, because $y_\infty(o)$ is the smallest minimizer of $V(x, o)$ and $y_\infty(o) > \bar{y}(o)$ by assumption. But this contradicts $V_t(\bar{y}(o), o|T) \to V(\bar{y}(o), o)$.

For Part 3, take the limits of both sides of the functional equation for finite horizon problem given in equation (3) in Özer and Wei [1] with $K = 0$. The left hand side converges to $J(x, o)$. By Part 2, the smallest minimizer $y_t(o|T)$ converges to $y_\infty(o)$, therefore $\min_{x\leq y\leq x+Q} V_t(y, o|T) \to \min_{x\leq y\leq x+Q} V(y, o)$. □

Proof of Theorem 10. The total expected cost can be decomposed into two parts: fixed cost; and production and inventory cost $B(x, o|Y)$. Suppose the fixed cost were charged in every period regardless of the production decision, the maximum total fixed cost during infinite horizon would be $\frac{K}{1-\alpha}$. Therefore, $E \lim_{T\to\infty} \sum_{j=t}^{T} K\delta(y_j - x_j) + \alpha^{j-t} G_j(y_j) \leq \frac{K}{1-\alpha} + B(x, o|Y) = O(|x|^p)$. Note that $B(x, o|Y) = O(|x|^p)$ still holds because all-or-nothing policy is a subset of the admissible policies. Therefore, by an argument similar to that in the proof of Theorem 9, we can show that $\{J_t(x, o|T)\}$ is bounded above. From Theorem 8, We also know that $\{J_t(x, o|T)\}$ is increasing in T. Hence it converges to a limit $J(x, o)$.

For Part 2, note that $EJ(x - o^1 - Z, O) < \infty$ for all x, so $V(x, o)$ is well defined. Since each $J_t(x, o|T) \leq J(x, o)$, by the Lebesgue Convergence Theorem, $EJ_t(x - o^1 - Z, O|T) \to EJ(x - o^1 - Z, O)$, so $V_t(x, o|T) \to V(x, o)$, and $H_t(x, o|T) \to H(x, o)$. Since $H_t(x, o|T)$ is increasing, $H(x,o)$ is increasing. To prove that $\lim_{T\to\infty} s_t(o|T) = s_\infty(o)$, we define $\bar{s}(o) = \sup\{s_t(o|T)\}_{T=t}^{\infty}$ and use contradiction. If $s_\infty(o) < \bar{s}(o)$, then choose N such that $s_\infty(o) < s_t(o|N) < \bar{s}(o)$. By Theorem 8, $H_t(s_\infty(o), o|T) \leq H_t(s_\infty(o), o|N) < 0$ for $T \geq N$. Hence $H(s_\infty(o), o) < 0$, contradicting the definition of $s_\infty(o)$. If $s_\infty(o) > \bar{s}(o)$, we have $H(\bar{s}(o), o) < 0$ and $H_t(\bar{s}(o), o|T) \geq 0$ for all $T \geq t$. Hence $H(\bar{s}(o), o) - H_t(\bar{s}(o), o|T) < 0$, but this contradicts $H_t(\bar{s}(o), o|T) \to H(\bar{s}(o), o)$.

For Part 3, take the limits of both sides of equation (5) in Özer and Wei [1], the left hand side converges to $J(x,o)$. By Part 2, $s_t(o|T)$ converges to $s_\infty(o)$, therefore $\min\{K + V_t(x+Q, o|T), V_t(x, o|T)\} \to \min\{K + V(x+Q, o), V(x,o)\}$. □

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