This article was downloaded by: [2601:600:8380:ef60:4c4b:42da:1a80:26f2] On: 20 October 2025, At: 21:34
Publisher: Institute for Operations Research and the Management Sciences (INFORMS)
INFORMS is located in Maryland, USA

Operations Research

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

Bounds, Heuristics, and Approximations for Distribution Systems

Guillermo Gallego, Özalp Özer, Paul Zipkin,

To cite this article:
Guillermo Gallego, Özalp Özer, Paul Zipkin, (2007) Bounds, Heuristics, and Approximations for Distribution Systems. Operations Research 55(3):503-517. https://doi.org/10.1287/opre.1060.0373

Full terms and conditions of use: https://pubsonline.informs.org/Publications/Librarians-Portal/PubsOnLine-Terms-and-Conditions

This article may be used only for the purposes of research, teaching, and/or private study. Commercial use or systematic downloading (by robots or other automatic processes) is prohibited without explicit Publisher approval, unless otherwise noted. For more information, contact permissions@informs.org.

The Publisher does not warrant or guarantee the article's accuracy, completeness, merchantability, fitness for a particular purpose, or non-infringement. Descriptions of, or references to, products or publications, or inclusion of an advertisement in this article, neither constitutes nor implies a guarantee, endorsement, or support of claims made of that product, publication, or service.

Copyright © 2007, INFORMS

Please scroll down for article—it is on subsequent pages

With 12,500 members from nearly 90 countries, INFORMS is the largest international association of operations research (O.R.) and analytics professionals and students. INFORMS provides unique networking and learning opportunities for individual professionals, and organizations of all types and sizes, to better understand and use O.R. and analytics tools and methods to transform strategic visions and achieve better outcomes.
For more information on INFORMS, its publications, membership, or meetings visit http://www.informs.org

Operations Research
Vol. 55, No. 3, May–June 2007, pp. 503–517
ISSN 0030-364X | EISSN 1526-5463 | 07 | 5503 | 0503
DOI 10.1287/opre.1060.0373

Bounds, Heuristics, and Approximations for Distribution Systems

Guillermo Gallego
Columbia University, 500 West 120th Street, MC 4704, New York, New York 10027,
ggallego@ieor.columbia.edu

Özalp Özer
Stanford University, Terman 314, 380 Panama Way, Stanford, California 94305,
oozer@stanford.edu

Paul Zipkin
Fuqua School of Business, Duke University, Durham, North Carolina 27708,
paul.zipkin@duke.edu

This paper develops simple approximate methods to analyze a two-stage distribution system consisting of one warehouse and multiple retailers with stochastic demand. We consider local and central control schemes. The main ideas are based on relaxing and or decomposing the system into more manageable newsvendor-type subsystems. We also provide bounds on the optimal policy and the optimal expected cost. We show that one of the heuristics is asymptotically optimal in the number of retailers. These results provide practically useful techniques as well as insights into stock-positioning issues and the drivers of system performance.

Subject classifications: inventory, production: multi-item, echelon, stage, approximation, heuristics; uncertainty: stochastic; operating characteristics.
Area of review: Manufacturing, Service, and Supply Chain Operations.
History: Received June 2003; revisions received May 2005, February 2006, July 2006; accepted July 2006.

1. Introduction

Consider a two-level distribution system: Goods enter the system from an outside source and proceed first to the warehouse. The warehouse in turn supplies J retailers, where customer demands occur. Each such shipment requires a lead time, but no fixed cost. Demands are stochastic. Those that cannot be filled immediately are backlogged. There is an inventory-holding cost at each location and a backorder penalty cost at each retailer. Time is continuous, the horizon is infinite, the data are stationary, and the objective is to minimize total average cost.

We distinguish two modes of control, central and local. Under central control, all information flows to one point, where all decisions are made. Local control means that each location observes local information and makes decisions accordingly. However, even under local control, a single decision maker provides operating rules to all locations, which the locations then implement in real time. The locations do not have their own distinct objectives, as they do in contracting models (see Chen 2003). For both control modes, we focus on a class of simple replenishment policies, base-stock or one-for-one policies.

Clark and Scarf (1960) initiated the study of distribution systems under central control. They pointed out that an optimal policy, if it exists, is very complex. Since then, research has focused on the identification of heuristics that perform well. The literature on local control begins with Sherbrooke (1968) and has focused from the beginning on heuristics. Comprehensive reviews can be found in Axsäter (1993) for local control and Federgruen (1993) for central control.

Despite considerable progress over the years, the subject remains fragmented and opaque to nonexperts. The computational methods involved are intricate and require voluminous data. Little is known about the key factors that drive performance. The literatures on local and central control have rarely intersected. We do not even know which mode performs better, nor by how much. Central control certainly would be better, if we knew the optimal policy, but we do not. This paper takes a step at filling this gap and compares local and central control.

This paper presents fast, transparent, effective, and robust heuristics and approximations. Sheer speed is important because of the massive scale of many real systems. For example, General Motors' service-parts organization manages over four million stock-keeping units serving eight thousand dealers, and they reset stock allocations daily. Even smaller companies typically track hundreds of thousands of items. Some of these proposed methods require only aggregate data. All reveal important relationships between parameters and performance.

Under *local control*, each location monitors its own inventory position. The retailers order replenishments from the warehouse, using base-stock policies. The warehouse fills these requests on a first-come-first-served basis. The warehouse orders from the outside source, also using a base-stock policy.

Our heuristic for local control is called the *restriction-decomposition* (RD) heuristic. It applies three simple subheuristics and selects the best. The first subheuristic restricts the warehouse to hold no inventory, while the second uses the largest possible warehouse inventory. Between these extremes, the third subheuristic fixes warehouse inventory to an intermediate value by setting its safety stock tos zero. In an extensive numerical study (over 400 cases), the heuristic's cost is on average only 1.23% more than the best base-stock policy's cost.

This finding suggests that rather crude methods suffice to determine the warehouse's inventory. On the other hand, it is unlikely that still simpler rules will perform adequately. The combined heuristic works better than the individual subheuristics alone. It is *not* true, for instance, that no warehouse inventory always works well.

We show that the RD heuristic is asymptotically optimal in the number of retailers. Thus, it captures the behavior of the system even better for many retailers.

We also develop two approximations, the normal and maximal approximations. These are extensions of analogous approximations for single-location systems. Both utilize only limited information about retailer demands, not the whole distributions. The main purpose of these approximations is to predict the system's performance, and tests show that they perform this task very well.

Central control is a very different philosophy. However, we focus on base-stock replenishment policies, and it turns out that a central policy of this type is equivalent to a local one (Axsäter and Rosling 1993). Given these policies, the key difference between local and central control is the *allocation* rule at the warehouse. Under central control, the warehouse utilizes current information about inventory levels at the retailers.

To simplify the problem, we *relax* certain constraints on the control variables. This relaxation leads to a simpler problem, whose solution provides a lower bound on the optimal average cost. The relaxed problem has the form of a series system for which the optimal policy is known—an echelon base-stock policy. Combined with a simple, plausible allocation policy, this solution also provides a heuristic. In our numerical tests the percentage difference between the cost of this heuristic and the lower bound—the optimality gap—was on average 12.02%. For periodic-review systems, a similar relaxation approach is developed by Federgruen and Zipkin (1984); see also Aviv and Federgruen (2001) and Özer (2003). This approach is related to the "allocation assumption" of Eppen and Schrage (1981). See also Diks and Kok (1998).

We also searched directly for the best policy of this form. This class has just two parameters—the warehouse's echelon base-stock level and an aggregate base-stock level for the retailers. To help guide the search, we obtain bounds on these quantities. The result was an average optimality gap of 2.92%. Thus, most of the heuristic's optimality gap can be eliminated by finding better base-stock levels.

These findings show that the lower bound is quite accurate, and a policy of this type performs well. The particular policy selected by the relaxation heuristic, however, is adequate but no better.

Current trends in logistics point in opposite directions: Wal-Mart's cross-docking system, a central part of its overall strategy (Stalk et al. 1992), operates with little warehouse stock. On the other hand, Apple Computers consolidates inventories at distribution hubs in Ireland for Europe and in Shanghai for China. The recent resurgence of mail-order retailing (Dell, Lands' End, Amazon, Medco, etc.) also attests to the strategic value of centralized stocks (see Simchi-Levi et al. 2000 for other examples). The effectiveness of any stock-positioning strategy, of course, depends on the control policies used. Our methods and numerical results provide a foundation that sheds light on current developments in practice.

The numerical study also compares the system's performance under local and central control. We find that central control is generally superior, but the difference varies widely. The value of central control increases in the warehouse's lead time, its holding cost, and the retailers' penalty costs.

Note that the same model describes a multi-item production system with a common intermediate product. This interpretation underlies the literature on postponed product differentiation; see Lee et al. (1993) and Aviv and Federgruen (2001). The warehouse represents the differentiation point, and the retailers represent the end products. The value of risk pooling corresponds to the value of delayed differentiation. Our results apply to this setting as well.

The rest of this paper is organized as follows. In §2, we introduce notation and describe the system dynamics. In §3, we study local control. We provide some preliminaries, followed by the RD heuristic and the two approximations. In §4, we study central control. We provide heuristics and an approximation. In §5, we report the numerical study. First, we report on performance measures, such as the optimality gap and the computational requirements. Next, we provide insights on system design issues and the value of centralization. Section 6 extends the analysis to compound Poisson demand. In §7, we conclude and suggest directions for future research.

2. Distribution System

Consider a two-level distribution system. All items enter the system from an external supplier and proceed first to location $j = 0$, called the warehouse. The warehouse in turn supplies J retailers, where the customer demands occur, indexed by $j = 1, \ldots, J$. Shipments from the external supplier arrive at the warehouse after time L_0. Shipments arrive at retailer j after time L_j. The retailers satisfy the customer demand from on-hand inventory, if possible. Unsatisfied demand at retailer j is backordered at a linear penalty cost rate b_j. All locations are allowed to carry inventory. The local (installation) holding cost is h_j per unit at retailer j. Holding inventory at the retailer is more expensive than holding it at the warehouse $h_j \geqslant h_0$ for $j > 0$. This could be due to more expensive storage space, overhead, or the value added by transportation. On the other hand, inventory located closer to the customer enables a quick response, hence reducing the possibility of a backorder at each retailer. Demand at each retailer j follows a Poisson process $\{D_j(t),\ t \geqslant 0\}$ with rate λ_j, and these are independent across retailers.

The problem is where to locate the inventory and how to control the system so as to minimize the long-run average holding and penalty costs.

2.1. Cost

Let $E[\cdot]$ denote expectation, $V[\cdot]$ variance, and $[x]^+ = \max\{0, x\}$. The state of the system just after all decisions are made is summarized by:

For each retailer j,

I_j: on-hand inventory at retailer j,

B_j: backorders at retailer j,

$IN_j = I_j - B_j$: net inventory at retailer j,

IT_j: inventory in transit to retailer j,

$ITP_j = IT_j + IN_j$: inventory transit position of retailer j

$ITP_r = \sum_{j>0} ITP_j$: sum of inventory transit positions at the retailers,

IO_j: inventory on order by retailer j,

$IOP_j = IO_j + IN_j$: inventory order position of retailer j.

For the warehouse,

I_0: on-hand inventory at the warehouse,

B_0: backorders at the warehouse,

B_{0j}: warehouse backorders due to the orders from retailer j,

$IN_0 = I_0 + ITP_r$,

IT_0: inventory in transit to the warehouse,

$ITP_0 = IT_0 + IN_0$.

We use D_j to denote the lead time demand for location j in equilibrium, a generic random variable that has the distribution of $D_j(t, t + L_j] = D_j(t + L_j) - D_j(t)$. Note that $IO_j \geqslant IT_j$ for $j > 0$ because retailer j can be replenished only when the warehouse has inventory. Also, the order and transit positions are the same for the warehouse because it orders from an external supplier with ample stock. We use the superscript $^-$ to refer to states just before decisions. For example, ITP_j^- refers to the inventory transit position at retailer j before any decision is made.

Under any policy, the total average cost can be expressed as

$$h_0 E[I_0] + h_0 \sum_{j>0} E[IT_j] + \sum_{j>0} (h_j E[I_j] + b_j E[B_j]) \tag{1}$$

$$= H_0 E[IN_0] + \sum_{j>0} (H_j E[IN_j] + (b_j + h_j) E[B_j]), \tag{2}$$

where $H_j = h_j - h_0$ and $H_0 = h_0$ are the echelon holding costs. The first expression is based on local-cost accounting, and the second on echelon-cost accounting. We use c to denote the resulting cost under a local control policy and C for central control. For series systems, the cost and the optimal policy under local and central control are the same (Axsäter and Rosling 1993, Chen and Zheng 1994, Gallego and Zipkin 1999), but this is not so for distribution systems.

3. Local Control

Under local control, each location monitors its own inventory. Whenever the inventory-order position IOP_j at retailer j falls below the local base-stock level s_j, the retailer orders from the warehouse to raise IOP_j up to s_j. The sum of the retailers' orders constitutes the warehouse's demand process. This too is a Poisson process with rate $\lambda_0 = \sum_{j>0} \lambda_j$. The warehouse replenishes its own inventory from the outside supplier whenever its inventory-order position is below s_0. It satisfies the retailers' requests on a first-come-first-served basis. Note that information and control are decentralized or localized, in that each location sees its own demand and monitors its own inventory-order position. The exact analysis of this system is due to Simon (1971) and Graves (1985). See also Axsäter (1990).

3.1. Preliminaries

Following a top-down approach—that is, analyzing first the warehouse and then the retailers—we have

$$B_0 = [D_0 - s_0]^+, \tag{3}$$

$$I_0 = [s_0 - D_0]^+, \tag{4}$$

$$B_j = [B_{0j} + D_j - s_j]^+ \quad \text{for } j > 0, \tag{5}$$

$$I_j = [s_j - B_{0j} - D_j]^+ \quad \text{for } j > 0. \tag{6}$$

Here, B_{0j} and D_j are independent, and $(B_{0j} \mid B_0)$ is binomial with parameters B_0 and $\theta_j = \lambda_j / \lambda_0$. Given the s_j, one can compute the $E[I_j]$ and $E[B_j]$, and thus

$$c(s_0, s_1, \ldots, s_J) = h_0 E[I_0] + \sum_{j>0} c_j(s_0, s_j), \tag{7}$$

$$c_j(s_0, s_j) = h_j E[I_j] + b_j E[B_j]. \tag{8}$$

We omit the average holding cost of shipments in transit because it is the constant $h_0 \sum_{j>0} \lambda_j L_j$. Let $s^* = (s_j^*)_{j=0}^J$ denote the policy that achieves the minimum average cost c^*.

For fixed s_0, the total average cost in (7) separates into a constant, plus functions c_j of one variable each (s_j), each convex in its variable. This separation is quite useful computationally. On the other hand, the remaining problem is still not trivial. To compute $E[B_j]$ and $E[I_j]$ requires numerical convolution of B_{0j} and D_j. Also, the cost $c(s_0, s_1^*(s_0), \dots, s_J^*(s_0))$ is not convex in s_0. Finding the optimal s_0, therefore, requires an exhaustive search. To simplify this search, we later provide an upper bound on s_0^* (see Proposition 1).

We now present several heuristics and approximations. Some of these methods provide closed-form expressions using only the original problem data. They are easy to describe, compute, and implement. They also enable us to study the effect of system parameters on the optimal cost and policy.

3.2. The Restriction-Decomposition (RD) Heuristic

This approach restricts the policy space and then decomposes the resulting model into independent, single-location, newsvendor-type subsystems. The overall RD heuristic applies three subheuristics and selects the best. The first subheuristic restricts the warehouse to hold zero inventory, while the second uses maximal warehouse inventory. The third subheuristic uses an intermediate value, restricting the warehouse to zero safety stock.

3.2.1. Cross-Docking (CD) Subheuristic. The first subheuristic restricts s_0 to 0. We call this restriction *cross-docking*. This phrase refers to a warehouse-management approach that, among other things, utilizes little warehouse inventory. The warehouse operates as a repackaging or break-bulk center. For a general discussion of managerial advantages of cross-docking, see Rosenfield and Pendrock (1980). Our restricted system can be viewed as a stylized representation of this practical approach.

Thus, $I_0 = 0$ and $B_0 = D_0$. In this case, B_{0j} has the Poisson distribution with mean $L_0\lambda_j$. The average cost is

$$\begin{aligned} c(0,s_1,\dots,s_J) &= \sum_{j>0} c_j(0,s_j) \\ &= \sum_{j>0}\big(h_jE[s_j-B_{0j}-D_j]^+ \\ &\qquad + b_jE[B_{0j}+D_j-s_j]^+\big), \end{aligned}$$

where $B_{0j}+D_j$ is Poisson with mean $(L_0+L_j)\lambda_j$. Each retailer j operates as an independent newsvendor subsystem with total lead time L_0+L_j. Let $s_j^*(0)$ denote its optimal base-stock level.

3.2.2. Stock-Pooling (SP) Subheuristic. For any policy, from (5), we have $B_j \leqslant B_{0j} + [D_j - s_j]^+$ for all $j>0$, so

$$\begin{aligned} c(s_0,s_1,\dots,s_J) &= h_0E[I_0] + \sum_{j>0}(h_jE[I_j]+b_jE[B_j]) \\ &\leqslant h_0E[s_0-D_0]^+ \\ &\quad + \sum_{j>0}\big(h_jE[s_j-D_j]^+ + b_jE[B_{0j}] \\ &\qquad + b_jE[D_j-s_j]^+\big) \\ &= \Big(h_0E[s_0-D_0]^+ + \Big(\sum_{j>0}\theta_j b_j\Big)E[D_0-s_0]^+\Big) \\ &\quad + \sum_{j>0}(h_jE[s_j-D_j]^+ + b_jE[D_j-s_j]^+) \\ &= c_0(s_0) + \sum_{j>0} c_j(\infty, s_j), \qquad (9) \end{aligned}$$

where $c_0(s_0) = h_0E[s_0-D_0]^+ + b_0E[D_0-s_0]^+$ and $b_0 = \sum_{j>0}\theta_j b_j$. Each term in this expression is the cost of an independent newsvendor subsystem, one for each location, and we can optimize each one separately. Denote the solutions by $s_j^*(\infty)$ for $j>0$ and s_0^u.

Observe that this subheuristic uses minimal retailer inventories; the $s_j^*(\infty)$ assume ample warehouse stock. Also, $c_0(s_0)$ is precisely the remaining objective if all retailer stocks were fixed to zero. Therefore, this method uses maximal warehouse inventory. For this reason, we call this subheuristic *stock-pooling*. The model can be considered a stylized version of the practical method of pooling all inventory at the warehouse.

3.2.3. Zero-Safety-Stock (ZS) Subheuristic. The third subheuristic fixes $s_0 = E[D_0]$ and then optimizes over s_j, $j>0$. (If $E[D_0]$ is a fraction, round it down to the nearest integer.) Thus, the warehouse's safety stock is set to zero.[1] This entails solving J single-location problems. However, it requires convolutions for B_j and I_j, whereas the CD and SP subheuristics do not.

3.2.4. Properties of the RD Heuristic. Next, we present some useful bounds. We also show that the RD heuristic is asymptotically optimal.

Proposition 1. (1) $0 \leqslant s_0^* \leqslant s_0^u$.

(2) $s_j^*(\infty) \leqslant s_j^*(s_0^*) \leqslant s_j^*(0)$.

(3) $\sum_{j>0} c_j(\infty, s_j^*(\infty)) \leqslant c^* \leqslant \min\{c(0, s_1^*(0), \dots, s_J^*(0)), c_0(s_0^u) + \sum_{j>0} c_j(\infty, s_j^*(\infty))\}$.

The first two parts are due to Axsäter (1990). Intuitively, s_0^u is an upper bound on the optimal s_0^* because, in effect, it assumes no retailer inventories. Increasing s_j can only reduce the required warehouse stock. Similarly, $s_j^*(\infty)$ is a lower bound on s_j^* because it assumes infinite supply at the warehouse and ignores the backlog B_{0j}. The $s_j^*(0)$ is an upper bound for the opposite reason.

As for Part 3, $c_j(\infty, s_j^*(\infty))$ is a lower bound on retailer j's cost because it ignores warehouse backorders. The sum of these quantities is thus a lower bound on the optimal total cost c^*. The two quantities on the right-hand side are the costs of the solutions obtained by the first two subheuristics above, and so are upper bounds on c^*.

Let c^{RD} denote the cost of the RD heuristic. From Proposition 1,

$$\sum_{j>0} c_j(\infty, s_j^*(\infty)) \leqslant c^* \leqslant c^{\mathrm{RD}} \leqslant c_0(s_0^u) + \sum_{j>0} c_j(\infty, s_j^*(\infty)).$$

Consider a sequence of systems indexed by J, constructed as follows: There is an infinite sequence of retailers with fixed parameters, and system J is obtained by including the first J of them. The warehouse's parameters remain constant. Assume that the λ_j are bounded above, say $\lambda_j \leqslant \bar{\lambda}$. Also, the $c_j(\infty, s_j^*(\infty))$ are bounded below by $c_- > 0$.

Proposition 2. *Under the scheme above, the RD heuristic is asymptotically optimal in the number of retailers within the class of local base-stock policies.*

Using the bounds above,

$$\frac{c^{\mathrm{RD}}}{c^*} \leqslant \frac{c_0(s_0^u) + \sum_{j>0} c_j(\infty, s_j^*(\infty))}{\sum_{j>0} c_j(\infty, s_j^*(\infty))} = \frac{c_0(s_0^u)}{\sum_{j>0} c_j(\infty, s_j^*(\infty))} + 1$$
$$\leqslant \frac{\sqrt{h_0 b_0 L_0 (J\bar{\lambda})}}{J c_-} + 1 = \frac{1}{\sqrt{J}} \frac{\sqrt{h_0 b_0 L_0 \bar{\lambda}}}{c_-} + 1.$$

(The last inequality applies the distribution-free upper bound of Gallego and Moon 1993 to $c_0(s_0^u)$.) As J becomes large, this quantity goes to one.

3.3. Approximations

3.3.1. Normal Approximation (NA). Now we apply the normal approximation to our distribution system. Let ϕ denote the standard normal density function, Φ^0 the standard normal complementary cumulative distribution function, Φ^1 the standard normal loss function, and Φ^2 the standard normal second-order loss function, that is,

$$\Phi^0(z) = \int_z^\infty \phi(x)\,dx,$$
$$\Phi^1(z) = \int_z^\infty \Phi^0(x)\,dx = \int_z^\infty (x-z)\phi(x)\,dx$$
$$= -z\Phi^0(z) + \phi(z),$$
$$\Phi^2(z) = \int_z^\infty \Phi^1(x)\,dx = \frac{1}{2}[(z^2+1)\Phi^0(z) - z\phi(z)].$$

Note that, although there are no closed-form expressions for these integrals, all standard software packages (including MS Excel) today have a built-in function to compute $\Phi^0(z)$. The mean and also the variance of D_j, $j \geqslant 0$, are $L_j\lambda_j$. The standard normal approximation at the warehouse yields

$$E[B_0] = \Phi^1(z_0)\sqrt{L_0\lambda_0},$$
$$E[B_0(B_0-1)] = 2\Phi^2(z_0)L_0\lambda_0,$$
$$E[I_0] = \Phi^1(-z_0)\sqrt{L_0\lambda_0},$$

where $z_0 = (s_0 - L_0\lambda_0)/\sqrt{L_0\lambda_0}$. Moreover,

$$E[B_{0j}] = \theta_j E[B_0],$$
$$V[B_{0j}] = \theta_j(1-\theta_j)E[B_0] + \theta_j^2 V[B_0] \quad \text{for } j > 0. \tag{10}$$

We obtain the variance by the conditional variance formula. Now approximate each $B_{0j} + D_j$ by a normal distribution with mean and variance,

$$\hat{\mu}_j = E[B_{0j}] + L_j\lambda_j,$$
$$\hat{\sigma}_j^2 = V[B_{0j}] + L_j\lambda_j.$$

(Recall that B_{0j} and D_j are independent.) The normal approximation for retailer j yields

$$E[B_j] = \Phi^1(z_j)\hat{\sigma}_j,$$
$$E[I_j] = \Phi^1(-z_j)\hat{\sigma}_j,$$

where $z_j = (s_j - \hat{\mu}_j)/\hat{\sigma}_j$. We now have all the elements needed to evaluate the average cost c. Furthermore, if we set s_j optimally given s_0, that is, $s_j^*(s_0) = \hat{\mu}_j + z_j^*\hat{\sigma}_j$, then the average cost for retailer j is

$$c_j(s_0, s_j^*(s_0)) = (b_j + h_j)\phi(z_j^*)\hat{\sigma}_j,$$

where z_j^* solves $\Phi^0(z) = h_j/(b_j + h_j)$. This quantity depends on s_0 through $\hat{\sigma}_j$. The total average cost in (7), therefore, reduces to a function of one variable:

$$\min_{s_0} c(s_0, s_1^*(s_0), \ldots, s_J^*(s_0))$$
$$= \min_{s_0} \left\{ h_0 E[I_0] + \sum_{j>0} (b_j + h_j)\phi(z_j^*)\hat{\sigma}_j \right\}.$$

In general, this function is not convex. However, one can show that it has a unique local minimum. Thus, it is easy to optimize numerically. (We then truncate fractional values.)

Axsäter (2003) independently develops a similar but more intricate method for the more general case of (r, q) policies. Graves' (1985) two-moment approximation is similar in spirit, although different in detail. Neither reduces the problem to a one-dimensional optimization.

Next, we apply the normal approximation to the RD heuristic and the bounds of Proposition 1. The resulting approximate bounds are in closed form, and they provide insight into the system's performance. To do so, we approximate D_j and $B_{0j} + D_j$ by normal distributions with mean and variance $L_j\lambda_j$ and $(L_0 + L_j)\lambda_j$, respectively. We have

$$\sum_{j>0} (h_j + b_j)\phi(z_j^*)\sqrt{L_j\lambda_j}$$
$$\leqslant c^* \leqslant \min\left\{ \sum_{j>0} (h_j + b_j)\phi(z_j^*)\sqrt{(L_0 + L_j)\lambda_j}, \right.$$
$$\left. \sum_{j\geqslant 0} (h_j + b_j)\phi(z_j^*)\sqrt{L_j\lambda_j} \right\}.$$

(The inequalities, of course, should be interpreted as up to the approximation.)

To gain some insights into system performance, consider the case of identical retailers having parameters λ, h, b, and L. Here, $\theta_j = 1/J$. From (10),

$$\begin{aligned}\hat{\sigma}_j &= \sqrt{V[B_{0j}+D_j]}\\ &= \sqrt{\theta_j^2 V[B_0]+\theta_j(1-\theta_j)E[B_0]+\theta_j\lambda L}\\ &= \frac{1}{J}\sqrt{V[B_0]+(J-1)E[B_0]+J^2\lambda L}.\end{aligned}$$

The total average cost is

$$\begin{aligned}c(s_0, s_1^*(s_0),\ldots,s_J^*(s_0)) = h_0E[I_0]+(b+h)\phi(z^*)\\ \cdot\sqrt{V[B_0]+(J-1)E[B_0]+J^2\lambda L},\end{aligned}$$

where z^* solves $\Phi^0(z) = h/(b+h)$.

Let us examine the joint impact of s_0 and the number of retailers. First, consider $s_0 = 0$. If we fix the total mean demand λ_0, then the average cost is $(b+h)\cdot\phi(z^*)\sqrt{J(L_0+L)\lambda_0}$, which is proportional to $\sqrt{J}$. However, increasing s_0 increases $E[I_0]$ and reduces $V[B_0]$, whose effects are independent of the number of retailers. On the other hand, increasing s_0 reduces $E[B_0]$, and this has a coefficient of J. This suggests that the overall cost depends roughly on the square root of the number of retailers, but the strength of this dependence declines with s_0.

Hence, stock at the warehouse level has two purposes: (1) It serves to pool some of the retailers' demand uncertainty; this inventory can be used to fill an order from *any* retailer; and (2) it may be cheaper to carry inventory at the warehouse than at the retailers.

3.3.2. Maximal Approximation (MX). Consider the SP subheuristic. We can further approximate its cost using Gallego and Moon's (1993) distribution-free bound. The base-stock levels become

$$s_j^m = L_j\lambda_j + \frac{1}{2}\sqrt{L_j\lambda_j}\left(\sqrt{\frac{b_j}{h_j}}-\sqrt{\frac{h_j}{b_j}}\right)\quad \text{for } j\geqslant 0.$$

(Again, truncate any fractional values.) The corresponding cost is an upper bound on that of any system with lead time demand variances $L_j\lambda_j$:

$$c^* \leqslant \sum_{j\geqslant 0}\sqrt{h_jb_j}\sqrt{L_j\lambda_j}.$$

4. Central Control

Local control cannot fully utilize the benefit of risk pooling because it relies on history rather than on the current state to allocate stock to the retailers. Consider a scenario in which the warehouse runs out of stock and the first demand occurs at retailer j, while the next five demands occur at retailer k. At this point the warehouse receives a unit from its supplier. The warehouse allocates this unit to retailer j under local control, even though retailer k needs it more. It is here that we expect a central control mechanism to be beneficial. Under central control, a single decision maker collects the information about the system and decides (1) how much to *order* from an outside supplier to replenish the warehouse inventory, (2) how much to *withdraw* from the warehouse, and (3) how to *allocate* the withdrawn quantity to the retailers.

4.1. Analysis

The key control variables are the ITP_j. In those terms, the average cost in (2) can be written as

$$C = H_0E[IN_0]+\sum_{j>0}C_j(ITP_j), \tag{11}$$

$$C_j(y) = H_jE[y-D_j]+(b_j+h_j)E[y-D_j]^-. \tag{12}$$

The constraints are $ITP_j \geqslant ITP_j^-$ and $ITP_r - \sum_{j>0} ITP_j^- \leqslant I_0^-$. Let C^* denote the optimal cost under central control. Recall that the true optimal policy is unknown.

Note that the average cost includes the costs of shipments in transit. For the numerical study, we subtract the constant in-transit holding cost $h_0\sum_{j>0}E[IT_j]$ from (11), to be consistent with the local control case.

We present a heuristic, a direct search technique, and an approximation.

4.2. Heuristic and Direct Search

4.2.1. Relaxation-Based (RB) Heuristic. Using a relaxation approach, we construct a lower bound on the average cost (11). Based on this calculation, we propose a heuristic that yields a feasible solution and hence an upper bound on the optimal cost.

Imagine that at any time we can ship any amounts to the retailers, as long as the sum of the inventory-transit positions is equal to the amount available at the warehouse. We keep the constraint $\sum_j ITP_j = ITP_r$, but relax $ITP_j \geqslant ITP_j^-$. Under this scenario, because we can shift stock among the retailers at any time in the future, we can focus solely on the current cost rate. The cost rate under this relaxation is thus

$$H_0IN_0(t)+C_r(ITP_r),$$

where $C_r(x) = \{\min_{(y_1,\ldots,y_J)}\sum_{j>0}C_j(y_j) \text{ s.t. } \sum y_j = x\}$. This is the cost of a two-stage series system, and so the recursive algorithm for series systems (as in Chen and Zheng 1994 or Gallego and Zipkin 1999) minimizes it. Specifically, set S_r to minimize $C_r(y)$,

$$C_0(y) = H_0E[y-D_0]+EC_r(\min\{y-D_0, S_r\}), \tag{13}$$

and then set $S_0^{\#}$ to minimize $C_0(y)$. Observe that $S_r = \sum_{j>0}s_j^*$, where s_j^* minimizes $C_j(y)$. Also, $C_0(S_0^{\#})$ is a lower bound on C^* because it is obtained by relaxing the constraints $ITP_j \geqslant ITP_j^-$.

We use $S_0^\#$ from this calculation as the warehouse echelon base-stock level to decide how much to order from the supplier, and S_r as an aggregate retailer base-stock level to decide how much to withdraw from the warehouse. The allocation of the withdrawn quantity among the retailers is based on the solution of a problem similar to the one defining $C_r(x)$, but with the additional constraints $y_j \geqslant ITP_j^-$.

This approach is essentially that developed by Federgruen and Zipkin (1984), but they employ an additional approximation of the function C_r. We discuss that approximation in the following section.

Next, we obtain another lower bound, which is based on *aggregation.* Note that

$$\begin{aligned}
C &= H_0 E[IN_0] \\
&\quad + E\left[\sum_{j>0}\{H_j[ITP_j - D_j] + (b_j + h_j)[ITP_j - D_j]^-\}\right] \\
&\geqslant H_0 E[IN_0] + \min_{j>0}\{H_j\} E\left[\sum_{j>0} ITP_j - \sum_{j>0} D_j\right] \\
&\quad + \min_{j>0}\{b_j + h_j\} E\left[\sum_{j>0}[ITP_j - D_j]^-\right] \\
&\geqslant H_0 E[ITP_0 - D_0] + H_a E[ITP_r - D_a] \\
&\quad + (b_a + h_a) E[ITP_r - D_a]^-,
\end{aligned}$$

where $H_a = \min_{j>0}\{H_j\}$, $b_a = \min_{j>0}\{b_j\}$, $h_a = \min_{j>0}\{h_j\}$, and $D_a = \sum_{j>0} D_j$. The last inequality is due to $\sum_j \max(0, -x_j) \geqslant \max(0, -\sum_j x_j)$. This is precisely the cost rate of a two-stage series system with lead time demand D_0 at the upstream stage and D_a at the downstream stage. The second stage can be interpreted as an aggregate retailer. The optimal policy can be found by solving a series system as above to obtain echelon base-stock levels S_a and S_0^a. In our numerical study, we observed that the performance of this approach was inferior to that of the relaxation heuristic.

4.2.2. Direct Search (DS). Consider alternative policies of the same form, that is, using different base-stock levels (S_0, S_r) together with the allocation policy described in §4.2.1. To evaluate such a policy, one can simulate the system. Then, search directly over values of (S_0, S_r) and find the best one.

We can limit the search for the policy variables as follows. We first ignore the risk-pooling effect by restricting the warehouse to maintain a separate stock for each retailer. The warehouse orders from the source to replenish all these stocks. This restriction *decomposes* the system into J independent two-stage series systems. An echelon base-stock policy with base-stock levels (S_{0j}, S_j) is optimal for series system j. Of course, the total average cost is an upper bound on C^*. We conjecture that $\sum_{j>0} S_{0j}$ is an upper bound on the optimal echelon base-stock level S_0^* because this approach ignores the gains from risk pooling. Likewise, we conjecture that $\sum_{j>0} S_j$ is a lower bound on S_r^* because it ignores warehouse stockouts due to other retailers' orders. Furthermore, let S_j^u solve the newsvendor problem for retailer j with lead-time demand $D_{0j} + D_j$. We conjecture that $\sum_{j>0} S_j^u$ is an upper bound on S_r^*.

In sum, we search over values of $S_0 \in [0, \sum_{j>0} S_{0j}]$ and $S_r \in [\sum_{j>0} S_j, \sum_{j>0} S_j^u]$.

4.3. Normal Approximation (NA)

Next, we present a closed-form approximation of the function C_r above, and using that result, an easier way to compute a heuristic policy and a lower bound on the optimal cost. This approximation is developed in Federgruen and Zipkin (1984) for periodic review systems and summarized in Zipkin (2000). Assume equal retailer costs, i.e., $b_j = b$ and $h_j = h$, $j > 0$.

Approximate the lead-time demand D_j by a normal random variable with mean and variance $\lambda_j L_j$. Then, C_r becomes

$$C_r(y) = E[H(y - D_r) + (b + h)(y - D_r)^-],$$

where D_r is normal with mean $\mu_r = \sum_{j>0} \lambda_j L_j$ and standard deviation $\sigma_r = \sum_{j>0} \sqrt{\lambda_j L_j}$. The calculation can then proceed as above. Call the resulting solution (S_0^N, S_r^N). The resulting cost is a lower bound, up to the normal approximation, on the cost of every policy.

In the case $h_0 = h$, there is no incentive to carry inventory at the warehouse, and $S_0^N = 0$. The system reduces to a single-stage system with lead-time demand $\tilde{D} = D_0 + D_r$, having mean $\tilde{\mu} = \sum_{j\geqslant 0} \lambda_j L_j$ and variance $\tilde{\sigma}^2 = \lambda_0 L_0 + (\sum_{j>0} \sqrt{\lambda_j L_j})^2$. Thus, the optimal cost becomes

$$(b + h)\phi(z^*)\tilde{\sigma},$$

where z^* solves $\Phi(z) = b/(b + h)$.

In the case of identical retailers, with $\lambda_j = \lambda$ and $L_j = L$ for $j > 0$, $\tilde{\sigma}$ further simplifies to $\tilde{\sigma} = \sqrt{J\lambda(L_0 + JL)}$. Here, for fixed total lead time $L_0 + L$, $\tilde{\sigma}$ and hence the cost estimate decline as L_0 increases and L decreases. This reduction measures the value of postponement.

5. Numerical Results

This section tests the performance of the heuristics and approximations. Also, we provide insights into stock-positioning issues and discuss the value of centralization.

5.1. Performance Measures

For local control, we compare the exact solution to the heuristics. We measure the percentage error $\epsilon_i\% = (c_i - c^*)/c^*$ for $i \in \{\text{RD}, \text{NA}, \text{MX}\}$.

For the central control case, we measure the percentage difference between the lower bound and the upper bound $\epsilon_i\% = (\text{UB}_i - \text{LB})/\text{LB}$ for $i \in \{\text{RB}, \text{NA}, \text{DS}\}$. The upper bounds are obtained by simulation using 95% confidence intervals. Here, a small ϵ_i indicates both that the method is close to optimal and that the lower bound is accurate.

We test a total of 852 cases,[2] the details of which are described in the following subsection. We report the average and the standard deviation of the error measures over those cases as well as the required computation times. For some problem instances, we also report more specific details.

We are also interested in whether the approximations accurately predict the performance of the system. In other words, we investigate whether a change in a parameter, such as a lead time, changes the approximate cost in the same way as the true optimal cost. We use regression analysis for this purpose.

5.2. Description of Experiments

We perform three experiments. The first experiment tests cases with identical retailers. The second experiment covers nonidentical retailers; also, some parameters are generated randomly. The third experiment covers cases with some of the retailers having zero added value.

The first experiment includes 144 cases in which the retailers are identical. All have local holding cost $h_j = 1$ for $j > 0$. We consider all combinations of $L_0 = L_j \in \{0.10, 0.25\}$; $J \in \{2, 4, 8, 16, 32, 64\}$; $\lambda_0 \in \{16, 64\}$; $b_j \in \{9, 39\}$ (representing fill rates of 90% and 97.5%); and $h_0 \in \{0.3, 0.9\}$ (adding 30% and 90% of value at the warehouse), for a total of 96 cases. The remaining 48 instances use $L_0 \in \{0.1, 0.9\}$ with $L_j = 1 - L_0$ for $j > 0$ to compare systems with different degrees of risk pooling at the warehouse.

The second experiment includes 250 cases and allows for nonidentical retailers with randomly drawn parameters. In particular, we set $h_0 = 0.3$, $h_j = 1$, $J = \{2, 4, 8, 16, 32\}$, $L_0 = \{0.1, 0.25\}$, and $\lambda_0 = \{16, 32\}$. The parameters b_j and L_j are generated randomly, according to independent uniform distributions. We draw b_j from the interval $[9, 39]$ and L_j from $[0.1, 0.25]$. We consider all possible combinations of the other parameters, a total of 20 supercases. For each supercase, we generate 10 independent draws of b_j and L_j, for a total of 200 cases. In addition, we test the effect of a larger warehouse holding cost, fixing $h_0 = 0.9$, with $L_0 = 0.1$, $\lambda_0 = 16$, and $J = \{2, 4, 8, 16, 32\}$. This defines five more supercases; again, we generate 10 independent draws for b_j and L_j, for a total of 50 more cases. Note that unequal lead times have the same effect as unequal demand rates. Similarly, unequal penalty costs mean unequal holding-penalty-cost ratios.

The third experiment covers cases with some, but not all, retailers having zero added value, i.e., $h_j = h_0$ for some j, but $h_j > h_0$ for others. In particular, we fix $h_0 = 0.5$ and we set $h_j = 0.5$ for half the retailers. For the other half, we set $h_j = 1$. For the other parameters, we use $J = \{2, 4, 8\}$, $\lambda_j = \{16, 32\}$, $b_j = \{9, 39\}$, and $L_j = 0.5$. We tested all combinations of 12 cases. We also tested three additional cases with $J = \{2, 4, 8\}$ for which $b_j = 9$, $h_0 = 0.5$, $h_j = 0.5$ for half the retailers and $h_j = 1.5$ for the other half. In total, we test 15 cases.

Table 1. Summary statistics for the RD heuristic—identical retailers.

	Number of cases	Average (%)	σ_{RD} (%)		Number of cases	Average (%)	σ_{RD} (%)
$J = 2$	24	1.93	1.84	$\lambda_0 = 16$	96	1.30	1.77
$J = 64$	24	0.22	0.31	$\lambda_0 = 64$	48	0.96	0.82
$b = 9$	72	0.94	1.27	$h_0 = 0.3$	72	1.26	1.39
$b = 39$	72	1.43	1.72	$h_0 = 0.9$	72	1.11	1.64

5.3. Performance: Local Control

For the first experiment, the RD heuristic's average error is 1.18% (with standard deviation 1.53%). Table 1 provides some additional statistics. The performance of the heuristic is better for systems with shorter warehouse lead times L_0, lower penalty costs b, and higher retailer demand rates λ_j. It is relatively less sensitive to changes in the warehouse's holding cost. These observations are also supported by the asymptotic optimality of the RD heuristic. From the proof of Proposition 2, the heuristic converges to the optimal solution faster when L_0, b, and λ_j are small. All else being equal, increasing J reduces the error term for the heuristic, but the rate of reduction is problem specific.

The performance of the RD heuristic is better when the warehouse's lead time is shorter than the retailer lead times. The average error is 2.53% when $L_0 = 0.9$, and $L_j = 0.1$, but only 0.20% when $L_0 = 0.1$ and $L_j = 0.9$. When the warehouse lead time is longer, the system gains more from risk pooling. The RD heuristic evidently underestimates this effect.

The results for the second experiment are similar. The RD heuristic's average error is 1.29% (standard deviation is 1.09%). Table 2 reports additional statistics. Over the 250 cases, the ZS subheuristic yields the lowest cost in 194 cases, and the SP subheuristic in the remaining 56 cases. The CD sub-heuristic is never cheapest. (One can, however, construct problems with very short warehouse lead times for which the CD heuristic is best.)

The results for the third experiment are reported in Table 3. The RD heuristic's average error is 0.79% (standard deviation is 0.67%).

Table 2. Summary statistics for the RD heuristic—nonidentical retailers.

	Number of cases	Average (%)	σ_{RD} (%)		Number of cases	Average (%)	σ_{RD} (%)
$J = 2$	50	1.66	1.54	$\lambda_0 = 16$	150	1.21	1.11
$J = 32$	50	0.68	0.41	$\lambda_0 = 32$	100	1.40	1.06
$L_0 = 0.10$	150	0.75	0.80	$h_0 = 0.3$	200	1.53	1.04
$L_0 = 0.25$	100	2.07	0.97	$h_0 = 0.9$	50	0.30	0.63

Table 3. Performance of the RD heuristic when $h_0 = h_i = 0.5$ for $i \leqslant J/2$.

L_j	$h_j^\dagger$	b_j	λ_0	J	c_{RD}	$\epsilon_{\text{RD}}\%$	J	c_{RD}	$\epsilon_{\text{RD}}\%$	J	c_{RD}	$\epsilon_{\text{RD}}\%$
0.5	1.0	9	16	2	10.61	0.00	4	14.56	0.00	8	20.82	1.27
0.5	1.0	9	32	2	14.65	0.00	4	20.15	0.20	8	28.07	0.00
0.5	1.0	39	16	2	14.95	1.08	4	21.02	1.40	8	30.53	2.21
0.5	1.0	39	32	2	20.51	1.38	4	28.45	0.99	8	40.59	1.45
1.5	1.5	9	16	2	16.32	0.55	4	22.23	0.27	8	31.53	0.99

† for $j > J/2$.

In summary, the RD heuristic performs very well. Its subheuristics complement each other. Tables 4 and 5 report some of the problem instances.

The errors for the normal and maximal approximations are large for all three experiments. The average absolute error for the first experiment is 20.97% for the normal approximation and 30.2% for the maximal approximation. As expected, the normal approximation works poorly when the $L_j\lambda_j$ are small. Even so, the approximations are useful. We use regression to compare the approximate to the actual costs and report the R^2 in Table 6. We vary the parameters one at a time. Observe that R^2 is close to 1 for all factors. This suggests that both the NA and MX approximations can safely be used to investigate the impact of parameter changes on the system cost.[3]

The computation time required for the exact algorithm of §3.1 is increasing in the number of retailers, warehouse lead time, and the total demand rate at the warehouse. In particular, the exact algorithm requires on the order of J convolutions for each base-stock level s_0. The upper bound s_0^u is also proportional to J. Hence, the computational requirement for the algorithm is $O(J^2)$. The RD heuristic requires only one convolution per retailer (for the ZS subheuristic). Hence, the RD heuristic's complexity is $O(J)$. The computational time required for the approximations is negligible.

5.4. Performance: Central Control

In the first experiment, the average error for the RB heuristic is 12.86% (with standard deviation 25.18%). The errors are 24.15% (28.20%) for the NA and 2.84% (3.17%) for the DS method. The heuristic and the approximation perform better when the number of retailers is less than 32. In these cases, the average errors are 7.75% (10.33%) for the RB heuristic; 14.04% (16.04%) for the NA; and 3.07% (3.23%) for DS. Table 7 provides more results.

For the second experiment, the errors are 8.37% (5.25%) for the RB heuristic; 21.47% (14.66%) for the NA; and 2.85% (3.10%) for DS. Table 8 reports additional statistics.

From the summary statistics, we observe that the RB heuristic and the DS method perform better for (1) lower holding cost at the warehouse, (2) larger penalty costs and

Table 4. Performance of the RD heuristic when retailers are identical and $\lambda_0 = 16$, $b = 39$.

Parameters		Optimal		CD-subheuristic		SP-subheuristic		ZS-subheuristic		
	J	s_0^*/s_j^*	c^*	$s_j^*(0)$	c_{CD}	s_0^u/s_j^l	c_{SP}	s_0/s_j	c_{ZS}	$\epsilon_{\text{RD}}\%$
$L_0 = 0.1$	2	2/13	14.29	14	14.55	5/13	14.89	2/13	14.29	0.00
$L_j = 0.9$	4	2/8	20.76	8	21.38	5/8	21.42	2/8	20.76	0.00
$H_0 = 0.3$	8	2/5	30.39	5	31.20	5/5	30.99	2/5	30.39	0.00
	16	3/3	44.60	3	46.94	5/3	44.99	2/3	44.74	0.31
	32	3/2	65.89	2	68.90	5/2	66.24	2/2	66.13	0.37
	64	4/1	110.76	2	117.89	5/1	110.88	2/1	112.40	0.11
$L_0 = 0.1$	2	0/14	14.55	14	14.55	5/13	16.94	2/13	14.73	0.00
$L_j = 0.9$	4	1/8	20.99	8	21.38	5/8	23.46	2/8	21.20	0.99
$H_0 = 0.9$	8	1/5	30.70	5	31.20	5/5	33.04	2/5	30.83	0.42
	16	2/3	45.18	3	46.94	5/3	47.04	2/3	45.18	0.00
	32	2/2	66.57	2	68.90	5/2	68.29	2/2	66.57	0.00
	64	3/1	111.96	2	117.89	5/1	112.92	2/1	112.83	0.78
$L_0 = 0.9$	2	21/3	7.79	14	14.55	24/3	8.21	15/5	10.62	5.39
$L_j = 0.1$	4	21/2	10.28	8	21.38	24/2	10.76	15/3	13.51	4.67
$H_0 = 0.3$	8	22/1	15.20	5	31.20	24/1	15.42	15/2	18.03	1.45
	16	21/1	19.97	3	46.94	24/1	20.45	15/1	26.46	2.40
	32	19/1	34.04	2	68.90	24/1	34.89	15/1	36.43	2.49
	64	17/1	64.72	2	117.89	24/0	65.85	15/1	65.16	0.68
$L_0 = 0.9$	2	18/4	11.16	14	14.55	22/3	12.42	15/5	11.72	5.02
$L_j = 0.1$	4	19/2	13.63	8	21.38	22/2	14.94	15/3	14.61	7.19
$H_0 = 0.9$	8	17/2	18.66	5	31.20	22/1	19.79	15/2	19.13	2.52
	16	19/1	23.24	3	46.94	22/1	24.64	15/1	27.56	6.02
	32	17/1	36.49	2	68.90	22/1	38.96	15/1	37.53	2.85
	64	15/1	66.26	2	117.89	22/0	71.21	15/1	66.26	0.00

Table 5. Performance of the RD heuristic when retailers are not identical.

	Parameters		Optimal		RD heuristic			
$J=4$	$L_1/L_2/L_3/L_4$	$b_1/b_2/b_3/b_4$	$s_0/s_1/s_2/s_3/s_4$	c^*	Type	$s_0/s_1/s_2/s_3/s_4$	c_{RD}	%
$L_0=0.1$	0.12/0.16/0.14/0.18	19.99/20.12/27.90/15.90	3/2/2/2/2	8.61	ZS	2/2/2/2/2	8.92	3.60
$\lambda_0=16$	0.13/0.21/0.11/0.16	12.58/17.17/30.82/29.30	3/2/3/2/2	8.95	ZS	2/2/3/2/3	9.07	1.34
	0.13/0.24/0.21/0.11	34.89/37.95/32.67/19.26	3/2/3/3/2	10.06	ZS	2/2/3/3/2	10.34	2.78
	0.23/0.18/0.14/0.20	38.74/13.33/11.77/18.08	3/3/2/2/2	9.37	ZS	2/3/2/2/3	9.44	0.75
	0.13/0.17/0.22/0.14	12.18/24.30/09.78/36.90	3/2/2/2/2	9.05	ZS	2/2/3/2/3	9.16	1.22
	0.17/0.23/0.12/0.16	31.13/26.99/26.73/20.36	3/3/3/2/2	9.64	ZS	2/3/3/2/2	9.79	1.56
	0.21/0.23/0.24/0.16	29.80/31.32/24.97/15.00	3/3/3/3/2	10.26	ZS	2/3/3/3/2	10.39	1.27
	0.11/0.20/0.22/0.15	09.92/13.52/11.74/11.40	3/1/2/2/2	8.01	SP	4/1/2/2/2	8.11	1.25
	0.11/0.22/0.10/0.21	28.62/12.31/20.54/30.11	3/2/2/2/3	8.93	ZS	2/2/2/2/3	9.08	1.68
	0.19/0.25/0.13/0.15	25.13/18.84/21.28/31.24	3/3/3/2/2	9.76	ZS	2/3/3/2/3	9.85	0.92
$L_0=0.25$	0.20/0.18/0.23/0.18	13.64/29.78/16.32/30.78	11/4/4/4/4	13.73	ZS	9/4/4/4/4	14.22	3.57
$\lambda_0=32$	0.17/0.16/0.19/0.17	30.47/35.75/32.58/18.90	11/4/4/4/4	13.99	ZS	9/4/4/5/4	14.39	2.86
	0.10/0.15/0.14/0.24	22.16/20.47/31.29/37.22	11/3/3/4/5	13.48	ZS	9/3/4/4/5	13.77	2.15
	0.17/0.14/0.12/0.19	19.26/23.69/09.14/17.95	11/4/3/2/4	12.12	ZS	9/4/4/3/4	12.32	1.65
	0.14/0.14/0.25/0.18	10.23/20.14/11.60/35.76	11/3/3/4/4	12.42	ZS	9/3/3/4/4	12.95	4.27
	0.24/0.13/0.21/0.19	38.12/19.49/13.02/37.43	11/5/3/4/4	14.05	ZS	9/5/3/4/5	14.48	3.06
	0.21/0.13/0.23/0.10	16.02/20.10/25.48/25.22	11/4/3/5/3	12.80	ZS	9/4/3/5/3	13.16	2.81
	0.19/0.24/0.13/0.14	11.24/26.45/16.47/34.01	10/4/5/3/4	13.10	ZS	9/4/5/3/4	13.26	1.22
	0.16/0.14/0.20/0.17	38.21/10.35/25.18/22.25	11/4/3/4/4	12.99	ZS	9/4/3/5/4	13.37	2.93
	0.18/0.15/0.14/0.21	20.18/25.91/27.67/36.57	10/4/4/4/5	13.97	ZS	9/4/4/4/5	14.12	1.07

demand rates at the retailers, and (3) smaller number of retailers in the system. Tables 9 and 10 report some of the problem instances.

The results for the third experiment are shown in Table 11. The average gaps are 64.81% (61.17%) for the RB heuristic; 69.70% (56.40%) for the NA; and 4.99% (2.07%) for DS. These results indicate that, for this rather special type of system, the RB heuristic's performance deteriorates significantly. Dogru et al. (2005) also observe that the RB heuristic performs poorly for this special type of system. The DS method still performs well, although not quite as well for the other cases.

The time required to run each simulation was less than one minute on a Pentium III processor. The time required for the DS method is of course longer because it entails several simulations. The longest time among all problem instances was less than 30 minutes.

5.5. Insights

The numerical studies reveal many interesting insights about system behavior. We discuss these in three groups. The first group of observations is based on the local control case and the second on the central control case. Finally, we compare the two control strategies.

5.5.1. Cost of Cross Docking Under Local Control. Consider the special case where the warehouse cannot or does not hold inventory. We investigate how a cross-docking strategy affects the system performance under local control. The CD subheuristic of §3.2 is a stylized model of this strategy. The impact of cross-docking can be measured by the percentage difference in costs between this subheuristic and the optimal policy. In Figure 1(a), we fix $L_0+L_j=1$ and change L_0. We observe that the percentage

Table 6. Performance of the NA and MX approximations under local control for $\lambda_0=16$.

J	c^*	c_{NA}	c_{MX}	L_0	c^*	c_{NA}	c_{MX}	b	c^*	c_{NA}	c_{MX}	h_0	c^*	c_{NA}	c_{MX}
2	8.94	7.51	24.51	0.1	14.29	12.98	37.84	1	3.07	3.27	4.49	0.1	8.23	6.67	22.86
4	12.95	10.27	31.82	0.3	13.30	11.91	37.05	9	6.95	6.09	13.47	0.3	9.37	7.70	28.04
8	18.45	14.20	42.17	0.5	12.11	10.55	34.66	19	8.07	6.94	19.57	0.5	10.29	8.48	31.60
10	20.61	15.78	46.34	0.6	11.32	9.75	32.94	39	9.37	7.70	28.04	0.6	10.74	8.81	33.11
16	30.10	19.74	56.80	0.7	10.33	8.81	30.80	99	10.80	8.58	44.67	0.7	11.11	9.11	34.49
32	38.89	27.56	77.50	0.8	9.37	7.70	28.04	199	11.78	9.19	63.33	0.8	11.47	9.39	35.78
64	65.91	38.60	106.76	0.9	7.79	6.24	24.15	399	12.78	9.76	89.68	0.9	11.83	9.64	37.00
	$R^2=$	98.51%	98.53%		$R^2=$	99.89%	98.35%		$R^2=$	99.70%	80.52%		$R^2=$	99.83%	99.69%
	$L_0=L_j=0.25$, $b=39$, $h_0=0.3$				$J=2$, $L_0=0.8$, $L_j=0.2$, $b=39$, $h_0=0.3$				$J=2$, $L_0=0.8$, $L_j=0.2$, $h_0=0.3$				$J=2$, $L_0=0.8$, $L_j=0.2$, $b=39$, $h_0=0.3$		

Table 7. Summary statistics for central control methods—identical retailers.

	Number of cases	RB (%)	NA (%)	DS (%)		Number of cases	RB (%)	NA (%)	DS (%)
$J=2$	24	2.73	5.22	1.57	$\lambda_0=16$	96	16.17	29.53	3.00
$J=32$	24	10.67	37.28	3.75	$\lambda_0=64$	48	6.23	13.39	2.53
$b=9$	72	13.85	19.32	3.22	$h_0=0.3$	72	6.16	21.14	1.43
$b=39$	72	11.87	28.98	2.48	$h_0=0.9$	72	19.56	27.16	4.26

Table 8. Summary statistics for central control methods—nonidentical retailers.

	Number of cases	RB (%)	NA (%)	DS (%)		Number of cases	RB (%)	NA (%)	DS (%)
$J=2$	50	3.10	7.67	2.13	$\lambda_0=16$	150	9.65	24.32	3.76
$J=32$	50	10.49	41.55	2.52	$\lambda_0=32$	100	6.46	17.18	1.58
$L_0=0.10$	150	9.20	23.04	3.59	$h_0=0.3$	200	6.97	20.01	1.85
$L_0=0.25$	100	7.13	19.11	1.84	$h_0=0.9$	50	13.97	27.29	7.05

difference increases with the warehouse's replenishment lead time and the number of retailers. The system sacrifices larger gains due to risk pooling by avoiding inventory at the warehouse. On the other hand, an increase in warehouse holding cost reduces the percent difference, as illustrated in Figure 1(b). Note that even when the retailer holding costs are zero, the system gains from holding inventory at the warehouse. In conclusion, cross-docking may be a viable strategy when it is expensive to carry inventory at the warehouse, the penalty cost is high at the retailers, and the warehouse lead time is short relative to the retailer lead times.

5.5.2. Value of Safety Stock Under Local Control. To investigate the value of safety stock at the warehouse under local control, we plot the percentage difference between the ZS subheuristic and the optimal solution with respect to supplier-to-warehouse lead time and holding cost in Figures 2(a) and 2(b), respectively. This value increases with the lead time. Comparing this figure with Figure 1(a), we also observe that a large portion of the risk-pooling benefit can be realized by setting $s_0=E[D_0]$.

In Figure 2(b), we observe that the safety-stock value is convex with respect to the warehouse holding cost. This convexity is intuitive. For low holding cost, the optimal

Table 9. Performance of the NA, RB, and DS for central control—identical retailers.

	$L_0=0.1,\ L_j=0.9$								
	NA approximation			RB heuristic			DS		
J	S_0/S_0^N	C_{NA}	%	$S_0^\#/S_r$	C_{RB}	%	S_0^*/S_r^*	C^*	%
				$b=9,\ \lambda_0=16,\ H_0=0.3$					
2	23/22	10.45±0.009	1.08	23/22	10.42±0.020	0.78	24/22	10.34±0.020	0.02
4	26/26	15.24±0.014	2.74	26/28	15.41±0.034	3.86	26 24	14.88±0.019	0.28
8	30/30	22.52±0.019	3.63	33/32	21.90±0.010	0.77	32/30	21.81±0.106	0.37
16	36/37	32.82±0.049	6.45	35/32	30.85±0.025	0.04	32/47	31.71±0.139	2.84
32	44/46	50.24±0.097	8.61	35/64	49.77±0.082	7.59	32/43	49.17±0.567	6.28
64	55/59	76.50±0.074	17.41	66/64	65.18±0.036	0.04	64/64	66.16±0.057	1.54
				$b=9,\ \lambda_0=16,\ H_0=0.9$					
2	23/27	10.59±0.008	2.29	23/28	10.52±0.025	1.68	24/28	10.35±0.037	0.01
4	26/32	15.24±0.014	2.74	26/36	15.41±0.034	3.86	27/32	14.95±0.017	0.74
8	30/39	22.52±0.019	3.62	32/48	21.97±0.016	1.10	32/40	21.75±0.024	0.09
16	35/50	32.73±0.046	5.88	34/64	32.61±0.036	5.50	32/39	31.99±0.079	3.50
32	43/64	50.14±0.098	8.32	35/96	49.77±0.082	7.52	32/49	49.39±0.156	6.69
64	54/85	77.63±0.093	18.50	66/128	67.65±0.043	3.27	64/78	65.83±0.010	0.49
				$b=39,\ \lambda_0=16,\ H_0=0.3$					
2	27/26	14.47±0.044	1.71	28/26	14.43±0.070	1.39	28/26	14.25±0.012	0.11
4	31/30	21.65±0.064	4.83	33/32	20.99±0.106	1.62	34/33	20.76±0.028	0.53
8	37/37	33.51±0.197	10.51	42/40	30.53±0.040	0.66	40/38	30.55±0.142	0.74
16	46/46	50.28±0.387	12.81	51/48	44.84±0.074	0.61	48/58	46.03±0.501	3.28
32	58/60	83.14±0.939	26.24	67/64	65.87±0.158	0.02	67/64	65.99±0.451	0.20
64	76/78	119.30±1.233	8.18	69/128	119.63±0.910	8.48	72/67	110.38±0.894	0.09
				$b=39,\ \lambda_0=16,\ H_0=0.9$					
2	27/29	14.68±0.048	2.80	28/32	14.58±0.065	2.15	28/34	14.33±0.011	0.35
4	31/36	21.67±0.051	4.71	33/40	21.43±0.131	3.54	32/35	20.99±0.166	1.44
8	37/44	33.51±0.197	10.11	41/56	31.35±0.063	2.99	40/49	30.61±0.052	0.57
16	46/57	50.28±0.387	12.28	50/64	47.24±0.123	5.48	48/73	45.68±0.090	2.01
32	58/75	83.14±0.939	25.48	66/96	69.25±0.243	4.52	64/89	67.80±0.840	2.33
64	75/99	119.51±1.408	8.37	69/128	119.63±0.910	8.48	67/64	110.58±0.067	0.28

Table 10. Performance of the NA, RB, and DS for central control—nonidentical retailers.

	Parameters		NA approximation		RB heuristic		DS	
$J=4$	$L_1/L_2/L_3/L_4$	$b_1/b_2/b_3/b_4$	C_{NA}	%	C_{RB}	%	C^*	%
$L_0=0.1$	0.13/0.21/0.23/0.16	24/35.7/9.8/38.8	15.93±0.009	25.36	13.15±0.005	3.49	13.15±0.005	3.49
$\lambda_0=32$	0.19/0.11/0.18/0.13	34.3/27.8/28.7/14.9	13.99±0.009	16.46	12.24±0.002	1.84	12.13±0.005	0.95
$h_0=0.3$	0.23/0.12/0.12/0.21	18.4/37.2/17.6/19.1	12.90±0.003	6.72	12.79±0.006	5.84	12.15±0.02	0.58
	0.12/0.21/0.22/0.21	27/31.4/16.6/13.3	14.24±0.003	13.10	13.07±0.004	3.79	13.07±0.004	3.79
	0.10/0.11/0.22/0.23	15.3/12.5/25.6/9.4	12.24±0.005	11.74	11.53±0.002	5.23	11.53±0.002	5.23
	0.12/0.17/0.21/0.20	25.3/11.2/22.1/15.1	13.14±0.007	12.22	12.03±0.002	2.71	12.03±0.002	2.71
	0.20/0.14/0.17/0.14	26.3/25/27.9/13.8	13.77±0.009	13.50	12.83±0.003	5.72	12.47±0.021	2.75
	0.18/0.24/0.20/0.24	14.7/19.1/14.4/38.9	14.29±0.006	7.77	13.71±0.004	3.36	13.49±0.021	1.75
	0.17/0.25/0.12/0.19	11.8/22.1/36.9/10.5	13.38±0.010	12.77	12.47±0.001	5.13	12.09±0.023	1.96
	0.23/0.14/0.13/0.22	21.3/15.1/27.8/27.1	13.95±0.006	10.40	13.19±0.005	4.35	12.79±0.006	1.20
$L_0=0.25$	0.17/0.11/0.21/0.19	27.6/29.7/33.1/13.5	14.81±0.008	13.78	13.36±0.004	2.69	13.36±0.004	2.69
$\lambda_0=32$	0.19/0.23/0.24/0.19	30.8/10.3/29/38.3	16.30±0.011	15.21	14.42±0.003	1.93	14.27±0.047	0.91
$h_0=0.3$	0.15/0.18/0.15/0.13	12.3/35.1/34.5/31.3	15.33±0.016	20.86	13.10±0.005	3.34	12.77±0.021	0.73
	0.12/0.15/0.11/0.11	28.2/33.6/25.4/22.4	13.56±0.002	14.85	12.16±0.003	2.94	11.91±0.018	0.82
	0.16/0.14/0.17/0.18	13.6/18.7/31.1/18.4	13.33±0.004	8.90	12.50±0.002	2.11	12.50±0.002	2.11
	0.22/0.24/0.23/0.21	18/37.3/12.8/11	14.55±0.008	6.72	13.96±0.005	2.38	13.96±0.005	2.38
	0.22/0.18/0.19/0.24	11.2/35.3/28.6/18.7	15.06±0.013	10.20	14.06±0.005	2.93	14.06±0.005	2.93
	0.12/0.18/0.13/0.14	36.6/25.5/28.9/12.4	14.25±0.008	17.29	12.71±0.004	4.62	12.44±0.007	2.40
	0.17/0.16/0.18/0.22	24.3/20.5/29.6/25	14.30±0.009	6.54	14.08±0.002	4.93	13.79±0.002	2.78
	0.19/0.16/0.10/0.21	12/27.7/34.9/23.7	14.11±0.005	12.38	12.97±0.008	3.31	12.92±0.028	2.91

safety stock is large, and there is a large penalty for using zero instead. As the holding cost increases, the optimal safety stock decreases, and thus so does the penalty. When the holding cost is very high, however, it is optimal to carry negative safety stock, so it again becomes expensive to carry zero.

5.5.3. Cost of Cross-Docking Under Central Control. To investigate the cost of cross-docking under central control, we first obtain the best echelon base-stock levels (S_0^*, S_r^*) and the resulting average cost using direct search, as described in §4.2.2. Next, we reset $S_r = S_0^*$ and calculate the resulting cost C. This in effect shifts all the warehouse inventory to the retailers. Figure 3 plots the percentage difference in cost between the two. The cost of cross-docking is higher for longer warehouse lead times and smaller warehouse holding costs.

5.5.4. Local vs. Central Control. To quantify the benefit of central control, we compare the optimal local cost c^* to the cost of the best echelon base-stock policy from the direct search method. The percentage difference between the two (that is, $(c^* - C^*)/c^*$) measures the value of central control, that is, the cost reduction due to centralization.

In Table 12, we report some results for a system with $L_0 + L_j = 1$, $h_j = 1$, and $\lambda_0 = 16$. We observe that the value of central control increases with an increase in lead time and holding cost at the warehouse and the penalty cost at the retailers. For example, for a two-retailer system with $\lambda_0 = 16$, $L_0 = 0.8$, $b = 39$, and $h_0 = 0.9$, the best local policy is $(s_0^*, s_j^*) = (16, 5)$ with cost $c^* = 11.83$. For central control, direct search yields $(S_0, S_r) = (25, 11)$. The average cost of this policy is $C = 11.14 \pm 0.011$. This results in a cost reduction of $5.83\% = ((11.83 - 11.14)/11.83)$.

Recall that local and central control differ only in handling warehouse stockouts. In the last section of Table 12, by increasing the number of retailers, we reduce the mean demand at each retailer (because $\lambda_0 = 16$). For these cases, the warehouse seldom runs out of stock, and local control is nearly as effective as central control.

We also carried out a similar comparison using the parameters described in the third experiment with some, but not all, retailers having zero added value. We found some cases where local control works better. The average difference was −2.05%.

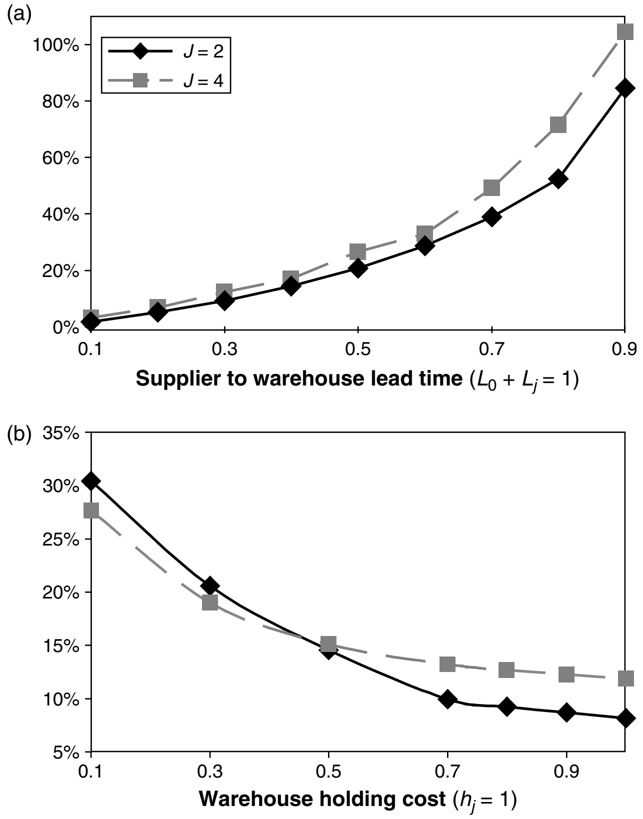
These observations suggest that local control can work well when (1) the warehouse holding cost is low, (2) the

Table 11. Performance of the NA, RB, and DS when $h_0 = h_i = 0.5$ for $i \leqslant J/2$.

L_j	$h_j^\dagger$	b_j	λ_0	J	$\epsilon_{\text{NA}}\%$	$\epsilon_{\text{RB}}\%$	$\epsilon_{\text{DS}}\%$	J	$\epsilon_{\text{NA}}\%$	$\epsilon_{\text{RB}}\%$	$\epsilon_{\text{DS}}\%$	J	$\epsilon_{\text{NA}}\%$	$\epsilon_{\text{RB}}\%$	$\epsilon_{\text{DS}}\%$
0.5	1.0	9	16	2	40.39	38.59	5.99	4	30.77	21.06	5.76	8	18.51	21.27	8.25
0.5	1.0	9	32	2	107.13	102.46	5.44	4	48.30	33.12	2.64	8	19.32	12.09	4.27
0.5	1.0	39	16	2	124.01	116.45	9.19	4	58.21	48.74	6.80	8	40.31	29.22	6.66
0.5	1.0	39	32	2	236.83	243.36	3.58	4	112.08	118.76	3.12	8	51.00	21.38	3.13
1.5	1.5	9	16	2	98.35	109.20	5.18	4	43.18	49.53	2.29	8	17.08	6.90	2.59

† for $j > J/2$.

Figure 1. Cost of cross-docking under local control when λ_0=16, b=9 for J=2, 4 w.r.t. (a) warehouse lead time when $L_0 + L_j = 1$, $h_j = 1$; (b) warehouse holding cost when $L_0 = L_j = 0.25$.



penalty costs are low, (3) the demand rates are small, and (4) the warehouse lead time is short relative to the retailer lead times. Otherwise, central control can improve system performance.

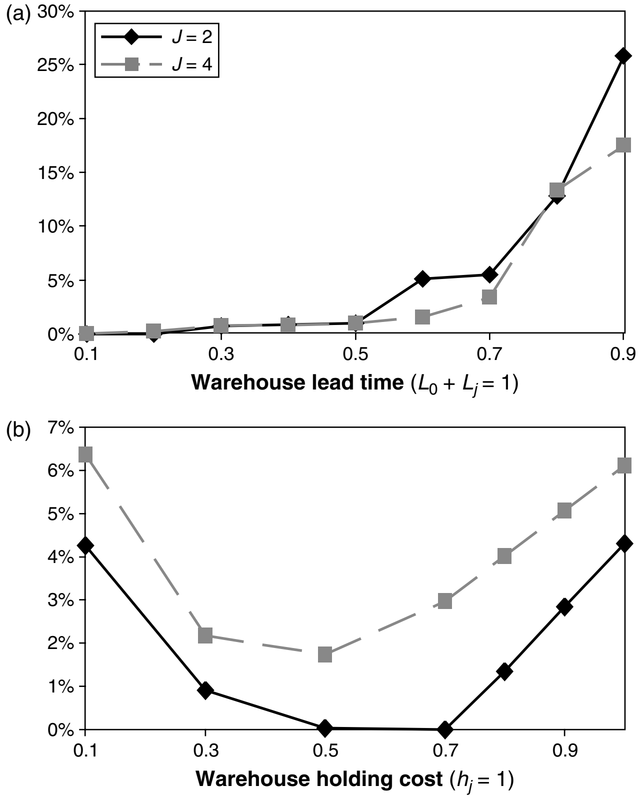
6. Extension: Compound Poisson Demands

Now suppose that demands at the retailers are independent compound Poisson processes, i.e., $D_j(t) = \sum_{k=1}^{N_j(t)} X_{kj}$, where $N_j(t)$ is the underlying Poisson process, and the order sizes X_{kj}, $k \geqslant 1$, are i.i.d. with mean μ_j and variance σ_j^2.

For central control, all the methods for the Poisson case in §4 extend immediately to compound Poisson demand. This is not the case for local control.

Consider now the local control case. The total demand at the warehouse, $D_0(t) = \sum_{j=1}^{J} D_j(t)$, is itself a compound Poisson process. Here, $D_0(t) = \sum_{k=1}^{N_0(t)} X_k$, where $N_0(t)$ is Poisson with rate $\lambda_0 = \sum_{j=1}^{J} \lambda_j$, and the demand size X_k is a mixture of the X_{kj} with mixing weights $\theta_j = \lambda_j / \lambda_0$, $j = \{1, \ldots, J\}$. Denote the mean and the variance of X_k by μ_0 and σ_0^2. It is not hard to determine the distribution of lead-time demand D_0. Using this characterization, we obtain B_0 and I_0 from (3) and (4).

Figure 2. Value of safety stock under local control when λ_0=16, b=9 for J=2, 4 w.r.t. (a) warehouse lead time when $L_0 + L_j = 1$, $h_j = 1$; (b) warehouse holding cost when $L_0 = L_j = 0.25$.



It is harder, however, to determine the B_{0j} and hence the B_j and I_j. (Axsäter and Zhang 1996 present an exact analysis.) Hence, we approximate B_{0j}. Let $\tau' = \inf\{t\colon D_0(t) \geqslant s_0\}$ and $\tau = \min(\tau', L_0)$. Then, B_0 is equal to the overshoot at time τ, namely, $D_0(\tau) - s_0$, plus the aggregate demand, say A_0, over the interval $(\tau, L_0]$. Let A_{0j} be the demand due to retailer j over the interval $(\tau, L_0]$. We find the mean and variance of A_{0j} in terms of those of A_0. We use these formulas to approximate the mean and variance of B_{0j} in terms of those of B_0. In essence, we approximate the distribution of the overshoot among the retailers in the same way that we distribute demand over the interval $(\tau, L_0]$.

Specifically, let N be the total number of arrivals in $(\tau, L_0]$. Then, $A_0 = \sum_{k=1}^{N} X_k$, and $A_{0j} = \sum_{k=1}^{N} f_j(X_k)$, where $f_j(X_k) = X_{kj}$ with probability θ_j and zero otherwise. Thus,

$$E[A_{0j}] = E[E[A_{0j} \mid N]] = \mu_j \theta_j E[N],$$

$$V[A_{0j}] = \mu_j^2 \theta_j^2 V[N] + \theta_j (EX_j^2 - \theta_j \mu_j^2) EN.$$

Figure 3. Cost of cross-docking under central control when $\lambda_0 = 16$, $b = 9$ for $J = 2, 4$ w.r.t. (a) warehouse lead time when $L_0 + L_j = 1$, $h_j = 1$ and (b) warehouse holding cost when $L_0 = 0.8$, $L_j = 0.2$.

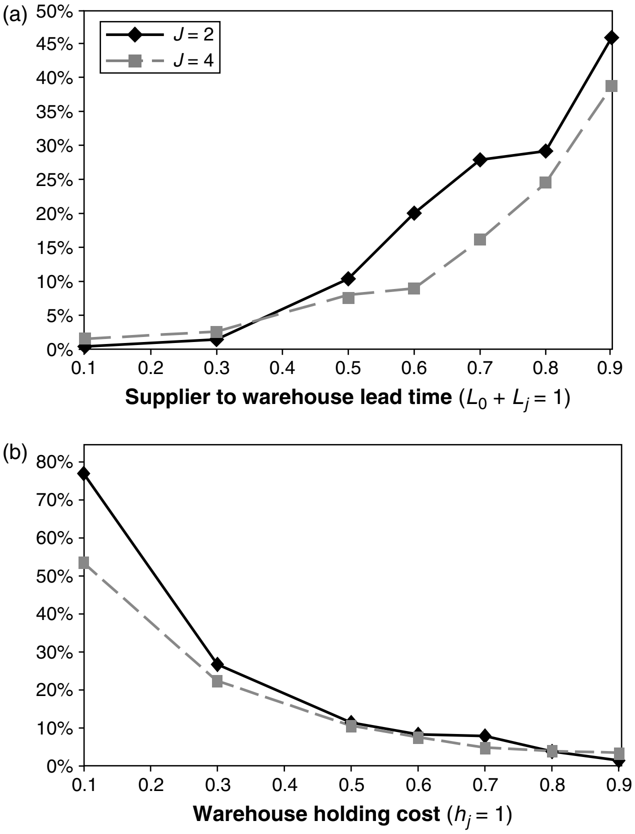


Table 12. Local vs. central control when $\lambda_0 = 16$, $h_j = 1$, and $L_0 + L_j = 1$.

L_0	c^*	C^*	%	h_0	c^*	C^*	%
0.1	14.29	14.11±0.10	1.30	0.1	8.23	8.02±0.02	2.52
0.3	13.30	13.16±0.02	1.02	0.3	9.37	9.13±0.02	2.60
0.5	12.11	11.81±0.06	2.40	0.5	10.29	10.02±0.01	2.67
0.6	11.32	11.15±0.02	1.53	0.6	10.74	10.36±0.01	3.28
0.7	10.33	10.04±0.01	2.89	0.7	11.11	10.72±0.01	3.55
0.8	9.37	8.94±0.02	3.39	0.8	11.47	11.03±0.02	3.63
0.9	7.79	7.78±0.01	2.21	0.9	11.83	11.14±0.01	5.87
$J = 2$, $h_0 = 0.3$, $b = 39$				$J = 2$, $L_0 = 0.8$, $b = 39$			

b	c^*	C^*	%	J	c^*	C^*	%
1	3.07	3.02 ± 0.00	1.52	2	9.37	9.10±0.00	2.88
9	6.95	6.77±0.05	2.60	4	12.59	12.43±0.02	1.27
19	8.07	7.92±0.02	1.82	5	14.52	14.19±0.07	2.27
39	9.37	9.04±0.02	3.58	8	17.70	17.35±0.03	1.97
99	10.8	10.52±0.04	2.67	10	20.52	20.21±0.01	1.52
199	11.78	11.35±0.04	3.70	16	27.35	27.26±0.04	0.30
399	12.78	12.18±0.09	4.68	32	37.20	37.18±0.07	0.06
$J = 2$, $L_0 = 0.9$, $h_0 = 0.3$				$L_0 = 0.8$, $h_0 = 0.3$, $b = 39$			

Table 13. Compound Poisson when $\mu_1 = 2$, $\mu_2 = 3$, $\mu_3 = 2$, $\mu_4 = 2$, $cv = 0.25$, $h_0 = 0.3$.

J	λ_0	L_0	L_j	b_j	c^*	RD	%
4	16	0.10	0.12/0.16/0.14/0.18	19.99/20.13/27.90/15.90	21.11	22.19	5.10
4	32	0.25	0.14/0.14/0.25/0.18	10.23/20.14/11.60/35.76	29.27	29.66	1.36
2	16	0.25	0.13/0.11	24.37/26.01	10.28	10.35	0.70
2	16	0.10	0.12/0.12	30.29/34.39	14.99	16.08	7.30
2	16	0.25	0.19/0.15	28.22/16.64	18.51	18.61	0.54
2	32	0.25	0.14/0.24	14.20/22.80	25.36	26.05	2.72

Also, $E[A_0] = \mu_0 EN$ and $V[A_0] = \mu_0^2 V[N] + \sigma_0^2 EN$. So,

$$E[A_{0j}] = \gamma_j E[A_0],$$

$$V[A_{0j}] = \gamma_j^2 V[A_0] + [\theta_j E[X_j^2] - \gamma_j^2 E[X_0^2]] \frac{E[A_0]}{\mu_0},$$

where $\gamma_j = \mu_j \theta_j / \sum_{i=1}^{J} \mu_i \theta_i$.

Replacing A_{0j} by B_{0j} and A_0 by B_0 in these equations yields the approximation of the moments of B_{0j}. This approximation is accurate when the distributions of the X_j are not too different, and the overshoot is a small part of the total backlog B_0.

We compared this method to the best such policy, focusing on the cases where the RD heuristic performed worst in the Poisson case. The worst cases correspond to $J = \{2, 4\}$. For the distribution of the demand sizes, we used normal distributions with means 2, 3, 2, and 2 and $cv = 0.25$. The mean for retailer 1 is 2, the mean for retailer 2 is 3, and so on. The same demand sizes were used for all experiments, except that only the first two retailers were used for experiments with $J = 2$. Table 13 reports the results. The experiments indicate that the approach performs well, resulting in an average error of 3.88% and a maximum error of 6.92%.

7. Conclusions

We establish simple bounds, heuristics, and approximation for distribution systems under local and central control. For local control, we show that the RD heuristic performs well and indeed is asymptotically optimal as the number of retailers increases. The normal and maximal approximations are easy to compute and are useful techniques for sensitivity analysis. For central control, we develop a relaxation scheme to obtain a lower bound on the optimal cost and a heuristic. The lower bound is quite accurate. The heuristic's performance is reasonably good.

We also quantify the value of central control. This comparison of local control and central control is, we believe, the most thorough to date. These comparisons also enable one to quantify system design issues. For example, the value of centralization is higher for a system with high holding cost and long lead time at the warehouse. Hence, a warehouse closer to its retailers gains more from centralization.

Endnotes

1. Schwarz (1981) proposes a similar idea but does not test it.

2. We run 409 cases for the local control problem, 409 cases for the central control problem, 28 cases to compare the two, and six cases for managerial insights.

3. We also tested the heuristics and approximations for a wider range of configurations with $h_0 = h_j = h$. We fixed the parameters $b = 1$, $\lambda_0 = 1$. The remaining parameters J, L_0, L_j, and h are generated randomly, according to independent uniform distributions. There are several cases with different parameter ranges. The following are the ranges for the base case: $J \in [2, 25]$, $h \in [0.0, 0.5]$, $L_0 \in [1, 25]$, $L_j \in [1, 25]$. In the other cases, some of the parameters (except J) have higher ranges: $h \in [0.5, 1.0]$, $L_0 \in [26, 50]$, $L_j \in [26, 50]$. We consider all possible combinations of base and high ranges. Thus, there are a total of eight cases. For each case, we generate 150 independent replications for a total of 1,200 problems.

The average error for the RD heuristic is 3.17%. For the NA, it is 5.19%. The error for the NA is even smaller, namely, 1.42% for the cases with high $L_0 \in [26, 50]$. The bound provided by the maximal approximation is not necessarily tight. Due to the space consideration, these results are omitted from the current paper. However, they are available from the authors.

Acknowledgments

The authors are thankful to the anonymous associate editor, the referees, Sven Axsäter, Kevin Shang, and Jeannette Song for their insightful comments and suggestions.

References

Aviv, Y., A. Federgruen. 2001. Design for postponement: A comprehensive characterization of its benefits under unknown demand distributions. *Oper. Res.* **49** 578–598.

Axsäter, S. 1990. Simple solution procedures for a class of two-echelon inventory problems. *Oper. Res.* **38** 64–69.

Axsäter, S. 1993. Continuous review policies for multi-level inventory systems with stochastic demand. S. Graves, A. Rinnooy Kan, P. Zipkin, eds. *Handbooks in Operations Research and Management Science*, Vol. 4. *Logistics of Production and Inventory*. North-Holland, Amsterdam, The Netherlands.

Axsäter, S. 2003. Approximate optimization of a two-level distribution system. *Internat. J. Production Econom.* **81–82** 545–553.

Axsäter, S., K. Rosling. 1993. Notes: Installation vs. echelon stock policies for multilevel inventory control. *Management Sci.* **39** 1274–1280.

Axsäter, S., W. Zhang. 1996. Recursive evaluation of order-up-to-S policies for two-echelon inventory systems with compound-Poisson demand. *Naval Res. Logist.* **43** 151–157.

Chen, F. 2003. Information sharing and supply chain coordination. S. Graves, T. de Kok, eds. *Handbooks in Operations Research and Management Science*, Vol. 11. *Supply Chain Management: Design, Coordination, and Operation.* North-Holland, Amsterdam, The Netherlands.

Chen, F., Y. Zheng. 1994. Lower bounds for multi-echelon stochastic inventory systems. *Management Sci.* **40** 1426–1443.

Clark, A., H. Scarf. 1960. Optimal policies for a multi-echelon inventory problem. *Management Sci.* **6** 475–490.

Diks, E. B., A. G. de Kok. 1998. Optimal control of a divergent multi-echelon inventory system. *Eur. J. Oper. Res.* **111** 75–97.

Dogru, M. K., A. G. de Kok, G. J. van Houtum. 2005. A numerical study on the effect of the balance assumption in one-warehouse multi-retailer inventory systems. Working paper, Eindhoven University of Technology, Eindhoven, The Netherlands.

Eppen, G., L. Schrage. 1981. Centralized ordering policies in a multi-warehouse system with leadtimes and random demand. L. B. Schwarz, ed. *Multi-Level Production/Inventory Control Systems: Theory and Practice.* North-Holland, Amsterdam, The Netherlands.

Federgruen, A. 1993. Centralized planning models for multi-echelon inventory systems under uncertainty. S. Graves, A. Rinnooy Kan, P. Zipkin, eds. *Handbooks in Operations Research and Management Science*, Vol. 4. *Logistics of Production and Inventory*. North-Holland, Amsterdam, The Netherlands.

Federgruen, A., P. Zipkin. 1984. Approximations of dynamic, multilocation production and inventory problems. *Management Sci.* **30** 69–84.

Gallego, G., I. Moon. 1993. The distribution free newsboy problem: Review and extensions. *J. Oper. Res. Soc.* **44** 825–834.

Gallego, G., P. Zipkin. 1999. Stock positioning and performance estimation for multi-stage production-transportation systems. *Manufacturing Service Oper. Management* **1** 77–88.

Graves, S. 1985. A multi-echelon inventory model for a repairable item with one-for-one replenishment. *Management Sci.* **31** 1247–1256.

Lee, H., C. Billington, B. Carter. 1993. Hewlett-Packard gains control of inventory and service through design for localization. *Interfaces* **23**(4) 1–11.

Özer, Ö. 2003. Replenishment strategies for distribution systems under advance demand information. *Management Sci.* **49** 255–272.

Rosenfield, D., M. Pendrock. 1980. The effects of warehouse configuration design on inventory levels and holding costs. *Sloan Management Rev.* **21**(4) 21–33.

Schwarz, L. B., ed. 1981. *Multi-Level Production/Inventory Control Systems: Theory and Practice*. North-Holland, Amsterdam, The Netherlands.

Sherbrooke, C. 1968. METRIC: A multi-echelon technique for recoverable item control. *Oper. Res.* **16** 122–141.

Simchi-Levi, D., P. Kaminsky, E. Simchi-Levi. 2000. *Designing and Managing Supply Chain: Concepts, Strategies and Case Studies*. McGraw-Hill, New York.

Simon, R. M. 1971. Stationary properties of a two-echelon inventory model for low demand items. *Oper. Res.* **19** 761–773.

Stalk, G., P. Evans, L. Shulman. 1992. Competing on capabilities: The new rules of corporate strategy. *Harvard Bus. Rev.* **70**(March–April) 57–69.

Zipkin, P. 2000. *Foundations of Inventory Management*. McGraw-Hill, New York.